Constitutive Models in Solid Mechanics

Course Notes — Technion, Fall 2025

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1. Constitutive Models in Solid Mechanics

Course Notes — Technion, Fall 2025

Welcome

These are the official course notes for **Constitutive Models in Solid Mechanics** (Technion, Fall 2025). They serve three purposes simultaneously:

What you're looking at	What it is
This website	Readable notes — study at your own pace, search, follow links
The **PDF download** button (top-right)	A printable, self-contained course textbook
The **Slides** links on each lecture page	RevealJS presentation used during lectures

The material is organized in three parts:

- **Part I** covers the mathematical and continuum mechanics foundations (tensors, kinematics, thermodynamics).
- **Part II** develops the main constitutive models: hyperelasticity, viscoelasticity, plasticity, and damage.
- **Part III** addresses parameter identification, verification/validation, and data-driven / surrogate approaches.

How to use these notes

In class (Zoom or in-person): The slides are projected. You don't need to copy everything — it's all here. Annotate your own copy, focus on understanding, ask questions.

After class: Work through the relevant chapter here. The notes contain more detail than the slides: derivations are written out in full, code examples can be expanded, and cross-references are active links.

For homework: Each assignment is in the Homework section. Submit via Moodle by the posted deadline.

Instructor

Shmuel Osovski Department of Mechanical Engineering, Technion Office: DK Building, Room 404 Email: shmuliko@technion.ac.il Office Hours: Mondays 15:00–16:00 or by appointment

Quick links

- Syllabus
- Homework 1
- Homework 2

- Homework 3
- Homework 4

2. Syllabus

2.1. Course information

Semester	Fall 2025
Lecture time	Mondays, 10:30–13:30
Location	LD Building, Room 440 (Zoom when remote)
Instructor	Shmuel Osovski
Office	DK Building, Room 404
Office hours	Mondays 15:00–16:00, or by appointment
Contact	shmuliko@technion.ac.il

2.2. Course description

This course provides a comprehensive introduction to the theory and application of constitutive models for describing the mechanical behavior of engineering materials. We cover the essential mathematical and continuum mechanics framework, then explore fundamental models for elasticity (hyperelasticity), plasticity (rate-independent and rate-dependent), and damage mechanics. A significant portion focuses on the **numerical implementation** of these models, typically in the context of the Finite Element Method (FEM). The course also addresses validation, verification, inverse modeling for parameter identification, and emerging data-driven approaches.

2.3. Prerequisites

- Undergraduate Solid Mechanics (stress, strain, Hooke's law)
- Multivariable calculus and differential equations
- Linear algebra (vectors, tensors/matrices)
- Basic programming (Python, MATLAB, Julia, or C++) — especially for the implementation part
- *Recommended but not required:* Introductory Continuum Mechanics, Introductory FEM

2.4. Learning objectives

Upon completion, students will be able to:

1. Apply tensor algebra and calculus for constitutive modeling, including eigenvalue problems and objective rates.
2. Analyze and implement kinematics, stress measures, conservation laws, and thermodynamic principles.
3. Formulate, interpret, and implement constitutive models: elastic, inelastic, rate-dependent, and damage.
4. Implement stress update algorithms (e.g., return mapping) and compute consistent tangent operators.
5. Design and execute inverse modeling processes to identify material parameters from experimental data.
6. Apply verification and validation methods.
7. Evaluate and implement data-driven and surrogate modeling approaches.

2.5. Course schedule (13 weeks)

Weeks	Topic	Lecture
1–2	Mathematical Preliminaries & Tensor Fundamentals	L01, L02
3	Continuum Mechanics Fundamentals & Kinematics	L03
4	Thermodynamic Framework & FEM Context	L04
5	Hyperelasticity & Viscoelasticity	L05, L06
6–9	Plasticity (theory, algorithms, rate-dependence)	L07, L08
10–11	Damage Mechanics & Coupled Problems	L09
11–12	Inverse Modeling & Parameter Identification	L10
13	Data-Driven & Surrogate Modeling	L11

2.6. Assessment

Component	Weight
Homework (4 sets)	60%
Final project — report	25%

Component	Weight
Final project — presentation	15%

Late policy: All assignments must be submitted electronically via Moodle by the posted deadline. No credit is given after the end of the semester.

Final project: Topics will be discussed in class. Students may choose from a provided list or propose their own (subject to approval). Presentations are in the last week. Reports are due two weeks after the semester ends.

2.7. Textbooks & references

Primary:

- Simo, J. C. & Hughes, T. J. R. *Computational Inelasticity.* Springer.
- de Souza Neto, E. A., Peric, D. & Owen, D. R. J. *Computational Methods for Plasticity: Theory and Applications.* Wiley.

Supplementary: Selected papers and book chapters distributed throughout the semester.

2.8. Academic integrity

All submitted work must be your own. Collaboration on concepts is encouraged; final submissions must be individually prepared.

For programming assignments: you may discuss algorithms with classmates, but your code must be written independently. Cite all external resources.

On LLM/AI tools: Use is permitted, but you take full responsibility for correctness. LLM-generated text left in submissions will be treated as plagiarism. Incorrect citations will also be treated as plagiarism. AI tools help, but they do not replace the learning process.

Part I.

Part I — Mathematical & Continuum Foundations

3. L01 — Mathematical Foundations

Vectors, Tensors, and Index Notation

3.1. Vectors and Vector Spaces

A **vector space** V over $\mathbb{R}$ is a set closed under addition and scalar multiplication satisfying the standard axioms: 1. Closure under addition: if $\mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} \in V$ 2. Associativity and commutativity of addition 3. Closure under scalar multiplication: if $\mathbf{u} \in V$ and $\alpha \in \mathbb{R}$ then $\alpha\mathbf{u} \in V$ 4. Distributive properties of scalar multiplication 5. Existence of additive identity (zero vector) and multiplicative identity

An n-dimensional space has exactly n linearly independent vectors.

> Note
>
> **Physical interpretation:** A vector is a mathematical object representing a direction and magnitude. In continuum mechanics, vectors describe positions ($\mathbf{x}$), displacements ($\mathbf{u}$), forces, and velocities. Unlike scalars (which depend only on magnitude), vectors transform in predictable ways under coordinate changes—a property essential for writing constitutive laws that do not depend on the observer's coordinate system.

3.1.1. Inner Product and Norms

The **inner product** (or dot product) of two vectors $\mathbf{u}, \mathbf{v} \in V$ is a scalar $\mathbf{u} \cdot \mathbf{v} \in \mathbb{R}$ satisfying:

1. **Symmetry:** $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. **Bilinearity:** $(\alpha\mathbf{u} + \beta\mathbf{v}) \cdot \mathbf{w} = \alpha(\mathbf{u} \cdot \mathbf{w}) + \beta(\mathbf{v} \cdot \mathbf{w})$ for $\alpha, \beta \in \mathbb{R}$
3. **Positive-definiteness:** $\mathbf{v} \cdot \mathbf{v} \geq 0$, with equality iff $\mathbf{v} = \mathbf{0}$
4. **Non-degeneracy:** if $\mathbf{v} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in V$, then $\mathbf{v} = \mathbf{0}$

The **Euclidean norm** (or length) of a vector is:

$$|\mathbf{v}| = \|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

Cauchy-Schwarz Inequality: For any $\mathbf{u}, \mathbf{v} \in V$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}|\,|\mathbf{v}|$$

Proof sketch: Consider $\mathbf{w}(\lambda) = \mathbf{u} - \lambda\mathbf{v}$ for $\lambda \in \mathbb{R}$. By positive-definiteness, $|\mathbf{w}(\lambda)|^2 = \mathbf{w}(\lambda) \cdot \mathbf{w}(\lambda) \geq 0$. Expanding and minimizing over λ yields the inequality. Equality holds iff $\mathbf{u}$ and $\mathbf{v}$ are parallel.

Triangle Inequality: $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ follows from Cauchy-Schwarz.

3.1.2. Angles, Orthogonality, and Orthonormal Bases

The **angle** θ between non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

Vectors are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

An **orthonormal basis** $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^3$ satisfies $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta. In an orthonormal basis, any vector can be uniquely written as:

$$\mathbf{v} = (v_1, v_2, v_3) = v_i \mathbf{e}_i$$

with components $v_i = \mathbf{v} \cdot \mathbf{e}_i$.

Worked example: Let $\mathbf{u} = (1, 2, 0)$ and $\mathbf{v} = (3, 0, 1)$ in the standard Cartesian basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

1. **Inner product:** $\mathbf{u} \cdot \mathbf{v} = 1 \cdot 3 + 2 \cdot 0 + 0 \cdot 1 = 3$.
2. **Norms:** $|\mathbf{u}| = \sqrt{1 + 4 + 0} = \sqrt{5}$, $|\mathbf{v}| = \sqrt{9 + 0 + 1} = \sqrt{10}$.
3. **Angle:** $\cos\theta = \frac{3}{\sqrt{5} \cdot \sqrt{10}} = \frac{3}{\sqrt{50}} = \frac{3}{5\sqrt{2}} \approx 0.424$, so $\theta \approx 65°$.
4. **Unit vector:** $\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{5}}(1, 2, 0)$.

3.2. Euclidean Space and Curvilinear Coordinates

In Euclidean space, we use **Cartesian coordinates** X^i (with $i = 1, 2, 3$) with orthonormal basis $\{\mathbf{E}_i\}$ where $\mathbf{E}_i \cdot \mathbf{E}_j = \delta_{ij}$.

For **general curvilinear coordinates** θ^i, we require a **smooth, invertible mapping** (diffeomorphism):

$$X^i = f^i(\theta^j) \quad \Leftrightarrow \quad \theta^j = g^j(X^i)$$

where g^j is the inverse of f^i. Both mappings must be continuous and differentiable. The position vector in Euclidean space is:

$$\mathbf{r}(\theta^1, \theta^2, \theta^3) = X^i(\theta^j)\, \mathbf{E}_i$$

3.2.1. Covariant (Tangent) Basis Vectors

As we move along a coordinate curve (say, varying θ^1 while holding θ^2, θ^3 fixed), the position vector changes. The **covariant basis vector** in direction i is the tangent vector to the θ^i coordinate curve:

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \theta^i} = \frac{\partial X^k}{\partial \theta^i}\mathbf{E}_k$$

These basis vectors are **not, in general, orthonormal** and **not unit vectors**. They form a local frame at each point in the coordinate system.

> **Note**
>
> **Physical interpretation:** In a deforming material (finite-strain kinematics), curvilinear coordinates attached to material particles move and stretch. The covariant basis vectors at a material point, when transported by the deformation, become the current basis vectors. This is why the distinction between covariant and contravariant components is essential: the former scale *with* basis changes (stretch), the latter scale *opposite* to basis changes (contract) to preserve geometric meaning.

3.2.2. Contravariant (Reciprocal) Basis Vectors

The **reciprocal (contravariant) basis vectors $\mathbf{g}^i$** are defined by the biorthogonality condition:

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$$

In three dimensions, the reciprocal basis can be constructed explicitly using the cross product:

$$\mathbf{g}^1 = \frac{\mathbf{g}_2 \times \mathbf{g}_3}{\mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3)}, \quad \mathbf{g}^2 = \frac{\mathbf{g}_3 \times \mathbf{g}_1}{\mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3)}, \quad \mathbf{g}^3 = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3)}$$

The denominator is the scalar triple product, which is non-zero if the basis is linearly independent.

3.3. The Metric Tensor

3.3.1. Definition and Geometric Meaning

The **metric tensor** (with covariant components) is defined by:

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$$

It is symmetric ($g_{ij} = g_{ji}$) and positive-definite (all eigenvalues are positive in a well-defined coordinate system).

Geometric meaning: - Diagonal elements g_{ii} (no sum) represent the squared length of the i-th basis vector: $|\mathbf{g}_i|^2 = g_{ii}$. - Off-diagonal elements g_{ij} (with $i \neq j$) encode the angle between $\mathbf{g}_i$ and $\mathbf{g}_j$: $\cos\theta_{ij} = \frac{g_{ij}}{\sqrt{g_{ii}g_{jj}}}$.

The **inverse metric tensor** g^{ij} satisfies:

$$g_{ik}g^{kj} = \delta_i^j$$

The arc length (infinitesimal distance) between two nearby points at θ^i and $\theta^i + d\theta^i$ is:

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \theta^i} d\theta^i \cdot \frac{\partial \mathbf{r}}{\partial \theta^j} d\theta^j = g_{ij}\, d\theta^i\, d\theta^j$$

In Cartesian coordinates, $g_{ij} = \delta_{ij}$ (unity metric).

Cylindrical coordinates (r, ϕ, z) with mapping $X^1 = r\cos\phi$, $X^2 = r\sin\phi$, $X^3 = z$:

Covariant basis vectors:

$$\mathbf{g}_r = \frac{\partial \mathbf{r}}{\partial r} = \cos\phi\,\mathbf{E}_1 + \sin\phi\,\mathbf{E}_2$$

$$\mathbf{g}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = -r\sin\phi\,\mathbf{E}_1 + r\cos\phi\,\mathbf{E}_2$$

$$\mathbf{g}_z = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{E}_3$$

Metric tensor components:

$$g_{rr} = 1, \quad g_{\phi\phi} = r^2, \quad g_{zz} = 1, \quad g_{r\phi} = g_{rz} = g_{\phi z} = 0$$

Inverse metric:

$$g^{rr} = 1, \quad g^{\phi\phi} = \frac{1}{r^2}, \quad g^{zz} = 1, \quad g^{r\phi} = g^{rz} = g^{\phi z} = 0$$

Arc length element: $(ds)^2 = dr^2 + r^2 d\phi^2 + dz^2$ ✓

Volume element: $dV = \sqrt{\det g_{ij}}\,dr\,d\phi\,dz = r\,dr\,d\phi\,dz$ (the Jacobian).

3.3.2. Spherical Coordinates (Optional Detail)

Spherical coordinates (R, θ, φ): $X^1 = R\sin\theta\cos\varphi$, $X^2 = R\sin\theta\sin\varphi$, $X^3 = R\cos\theta$.

Metric tensor:

$$g_{RR} = 1, \quad g_{\theta\theta} = R^2, \quad g_{\varphi\varphi} = R^2\sin^2\theta, \quad g_{R\theta} = g_{R\varphi} = g_{\theta\varphi} = 0$$

Volume element: $dV = R^2 \sin\theta\,dR\,d\theta\,d\varphi$.

3.4. Second-Order Tensors

A **second-order tensor $\mathbf{T}$** is a linear map $V \to V$. Component representation in a Cartesian basis:

$$\mathbf{T} = T_{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j$$

Key operations:

Operation	Expression
Transpose	$(T^T)_{ij} = T_{ji}$
Trace	$\operatorname{tr}\mathbf{T} = T_{ii}$
Determinant	$\det\mathbf{T}$
Double contraction	$\mathbf{A} : \mathbf{B} = A_{ij}B_{ij}$

Symmetric and Skew-symmetric decomposition:

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) = \mathbf{sym}(\mathbf{A}) + \mathbf{skew}(\mathbf{A})$$

3.5. Index Notation and the Summation Convention

Einstein summation: repeated index implies sum over 1, 2, 3.

$$a_i b_i \equiv \sum_{i=1}^{3} a_i b_i$$

Free indices appear once; dummy (summation) indices appear twice.

Examples:

$$(\mathbf{A}\mathbf{b})_i = A_{ij}b_j, \qquad (\mathbf{A}\mathbf{B})_{ij} = A_{ik}B_{kj}, \qquad \mathbf{A} : \mathbf{B} = A_{ij}B_{ij}.$$

The **Kronecker delta** δ_{ij} and **Levi-Civita symbol** ε_{ijk}:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \qquad \varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.$$

3.6. Principal Values and Directions

For a symmetric tensor $\mathbf{A}$, the eigenvalue problem:

$$\mathbf{A}\mathbf{n}^{(\alpha)} = \lambda^{(\alpha)}\mathbf{n}^{(\alpha)}, \qquad \alpha = 1, 2, 3.$$

The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0,$$

where I_1, I_2, I_3 are the **principal invariants**:

$$I_1 = \operatorname{tr}\mathbf{A}, \qquad I_2 = \tfrac{1}{2}[(I_1)^2 - \operatorname{tr}\mathbf{A}^2], \qquad I_3 = \det\mathbf{A}.$$

The spectral decomposition:

$$\mathbf{A} = \sum_{\alpha=1}^{3} \lambda^{(\alpha)}\,\mathbf{n}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)}.$$

3.7. Coordinate Transformations

Under an orthogonal transformation $\mathbf{Q}$ ($\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$):

$$\mathbf{v}^* = \mathbf{Q}\mathbf{v}, \qquad \mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T.$$

Invariants are unaffected by rotations.

3.8. Covariant and Contravariant Components

Any vector **u** can be written in two equivalent forms using the covariant and reciprocal bases:

$$\mathbf{u} = u^i \mathbf{g}_i = u_i \mathbf{g}^i$$

Contravariant components u^i (upper indices) are the coefficients when **u** is expanded in the **covariant basis** $\{\mathbf{g}_i\}$.

Covariant components u_i (lower indices) are the coefficients when **u** is expanded in the **reciprocal basis** $\{\mathbf{g}^i\}$.

Alternatively, covariant components can be obtained by **projection**:

$$u_i = \mathbf{u} \cdot \mathbf{g}_i$$

The two representations are related via the metric tensor (**raising and lowering indices**):

$$u_i = g_{ij} u^j \quad \text{(lower an index)} \quad \text{and} \quad u^i = g^{ij} u_j \quad \text{(raise an index)}$$

3.8.1. Geometric Intuition

In an **oblique basis** (non-orthonormal), the distinction is clear: - **Contravariant components** u^i answer the question: "How much of basis vector $\mathbf{g}_i$ do I need?" They scale **opposite** to how basis vectors scale. If we stretch the basis, contravariant components shrink. - **Covariant components** u_i are the **projections** of **u** onto the reciprocal vectors. They scale **with** the basis vectors.

In **Cartesian coordinates** where $\mathbf{g}_i = \mathbf{E}_i$ (orthonormal), the covariant and reciprocal bases are identical, so numerically $u_i = u^i$ and the distinction disappears. This is why introductory mechanics courses often ignore index placement!

Worked example (2D oblique basis): Let $\mathbf{g}_1 = (1, 0)$ and $\mathbf{g}_2 = (\cos\alpha, \sin\alpha)$ (two basis vectors at angle α).

Metric tensor: $g_{11} = 1$, $g_{22} = 1$, $g_{12} = \cos\alpha$.

Reciprocal basis (by formula, or by solving $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$):

$$\mathbf{g}^1 = (1, -\cot\alpha), \quad \mathbf{g}^2 = (0, \csc\alpha)$$

Consider the vector $\mathbf{u} = (1, 0)$ (horizontal unit vector in Cartesian space).

Contravariant components: Solve $\mathbf{u} = u^1\mathbf{g}_1 + u^2\mathbf{g}_2 = u^1(1, 0) + u^2(\cos\alpha, \sin\alpha)$. - From $1 = u^1 + u^2\cos\alpha$ and $0 = u^2\sin\alpha$, we get $u^2 = 0$ and $u^1 = 1$.

Covariant components: $u_i = \mathbf{u} \cdot \mathbf{g}_i$. - $u_1 = (1, 0) \cdot (1, 0) = 1$. - $u_2 = (1, 0) \cdot (\cos\alpha, \sin\alpha) = \cos\alpha$.

Verify via metric: $u_i = g_{ij}u^j$ gives $u_1 = g_{11} \cdot 1 = 1$ ✓ and $u_2 = g_{21} \cdot 1 = \cos\alpha$ ✓.

In this oblique system, the covariant representation $\mathbf{u} = 1 \cdot \mathbf{g}^1 + \cos\alpha \cdot \mathbf{g}^2$ is not as intuitive as the contravariant one; both are valid and equivalent.

3.9. Tensor Contractions

Outer Product ($\otimes$):

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$$

produces a second-order tensor (no summation).

Single Contraction ($\cdot$):

$$(\mathbf{A} \cdot \mathbf{v})_i = A_{ij}v^j$$

contracts one upper and one lower index, reducing order by 1.

Double Contraction (:):

$$\mathbf{A} : \mathbf{B} = A_{ij} B^{ij}$$

contracts two pairs of indices, producing a scalar result.

In curvilinear coordinates, use metric to align indices:

$$\mathbf{u} \cdot \mathbf{v} = u^i v_i = u_i v^i = g_{ij} u^i v^j$$

3.10. The Levi-Civita Symbol and Cross Product

3.10.1. Levi-Civita Permutation Symbol

The **Levi-Civita symbol** (or permutation symbol) ε_{ijk} in 3D is defined as:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if any two indices are equal} \end{cases}$$

Examples: $\varepsilon_{123} = +1$, $\varepsilon_{231} = +1$, $\varepsilon_{312} = +1$ (even), $\varepsilon_{213} = -1$, $\varepsilon_{132} = -1$, $\varepsilon_{321} = -1$ (odd), $\varepsilon_{112} = 0$.

Relation to the determinant: For a 3×3 matrix with entries A_{ij},

$$\det \mathbf{A} = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$$

(summing over all repeated indices).

3.10.2. Cross Product in Index and Direct Notation

The **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Index notation:

$$(\mathbf{u} \times \mathbf{v})_k = \varepsilon_{kij} u_i v_j$$

Geometric interpretation: - **Magnitude:** $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$, where $\theta \in [0, \pi]$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. - **Direction:** By the **right-hand rule**: point fingers along $\mathbf{u}$, curl them towards $\mathbf{v}$, thumb points in direction of $\mathbf{u} \times \mathbf{v}$. - **Geometric meaning:** The magnitude equals the **area of the parallelogram** spanned by $\mathbf{u}$ and $\mathbf{v}$. (The area of the triangle is half this.)

Properties: - **Anti-commutativity:** $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ - **Linearity:** $(\alpha\mathbf{u} + \beta\mathbf{v}) \times \mathbf{w} = \alpha(\mathbf{u} \times \mathbf{w}) + \beta(\mathbf{v} \times \mathbf{w})$ - **Orthogonality:** $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ - **Magnitude identity:** $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (follows from Cauchy-Schwarz)

3.10.3. Scalar Triple Product

The **scalar triple product** is the scalar:

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

Index notation:

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \varepsilon_{ijk} u_i v_j w_k$$

Geometric meaning: The absolute value $|[\mathbf{u}, \mathbf{v}, \mathbf{w}]|$ equals the **signed volume of the parallelepiped** (a 3D "slanted box") spanned by the three vectors. It is positive if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a right-handed system, negative if left-handed, and zero if they are coplanar.

3.10.4. The Epsilon-Delta Identity and BAC-CAB Rule

A fundamental identity relating the Levi-Civita symbol to the Kronecker delta is:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

This identity is used to prove the **BAC-CAB rule** (also called vector triple product rule):

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v})$$

Proof (index notation): Component i of LHS is

$$[\mathbf{u} \times (\mathbf{v} \times \mathbf{w})]_i = \varepsilon_{ijk}u_j(\mathbf{v} \times \mathbf{w})_k = \varepsilon_{ijk}u_j(\varepsilon_{klm}v_l w_m) = \varepsilon_{ijk}\varepsilon_{klm}u_j v_l w_m.$$

Using the epsilon-delta identity and relabeling, this simplifies to the RHS.

Worked example: Let $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$, $\mathbf{w} = (0, 0, 1)$ (standard basis).

Cross product: $\mathbf{v} \times \mathbf{w} = (1, 0, 0) = \mathbf{u}$.

Scalar triple product: $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{u} \cdot \mathbf{u} = 1$ (volume of unit cube).

BAC-CAB check: $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \times \mathbf{u} = \mathbf{0}$. By BAC-CAB, $\mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{v} \cdot 0 - \mathbf{w} \cdot 0 = \mathbf{0}$ ✓.

3.10.5. General Permutation Symbol in Curvilinear Coordinates

In curvilinear coordinates with metric tensor g_{ij}, the permutation symbol must be scaled by $\sqrt{\det g_{ij}}$ to account for the geometry:

$$\varepsilon_{ijk} = \sqrt{g}\, e_{ijk}$$

where $g = \det g_{ij}$ and e_{ijk} is the standard Cartesian Levi-Civita symbol. This ensures that the cross product formula and scalar triple product remain invariant (geometrically meaningful) under coordinate transformations.

3.11. Tensor Calculus: Gradient, Divergence, Curl

Gradient of a scalar $\phi(\mathbf{x})$:

$$(\nabla\phi)_i = \frac{\partial\phi}{\partial x^i} \quad \text{(covariant vector)}$$

Gradient of a vector $\mathbf{v}(\mathbf{x})$:

$$(\nabla\mathbf{v})_{ij} = \frac{\partial v_i}{\partial x^j} \quad \text{(second-order tensor)}$$

Divergence of a vector:

$$\nabla \cdot \mathbf{v} = \frac{\partial v^i}{\partial x^i} \quad \text{(scalar)}$$

Divergence of a tensor:

$$(\nabla \cdot \mathbf{T})_i = \frac{\partial T_{ij}}{\partial x^j} \quad \text{(vector)}$$

Curl of a vector:

$$(\nabla \times \mathbf{v})_k = \varepsilon_{kij}\frac{\partial v_i}{\partial x^j} \quad \text{(axial vector/pseudovector)}$$

In Cartesian coordinates, these simplifications hold. In curvilinear coordinates, Christoffel symbols modify the derivatives for covariance.

3.12. Tensor Invariants

For a second-order tensor $\mathbf{A}$, the principal invariants are:

$$I_1(\mathbf{A}) = \mathrm{tr}(\mathbf{A}) = A_{ii}$$

$$I_2(\mathbf{A}) = \frac{1}{2}\left[\mathrm{tr}(\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)\right] = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji})$$

$$I_3(\mathbf{A}) = \det(\mathbf{A})$$

These invariants remain unchanged under orthogonal coordinate transformations.

Deviatoric part: $\mathbf{A}' = \mathbf{A} - \frac{1}{3}\mathrm{tr}(\mathbf{A})\mathbf{I}$

Deviatoric invariants (e.g., $J_2 = \frac{1}{2}\mathbf{A}' : \mathbf{A}'$) are also invariant and commonly used in yield criteria.

3.13. Tensor Functions and Eigenvalue Decomposition

For a symmetric tensor $\mathbf{A}$ with spectral decomposition:

$$\mathbf{A} = \sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{n}_\alpha \otimes \mathbf{n}_\alpha$$

Any analytic tensor function $f(\mathbf{A})$ can be computed as:

$$f(\mathbf{A}) = \sum_{\alpha=1}^{3} f(\lambda_\alpha) \mathbf{n}_\alpha \otimes \mathbf{n}_\alpha$$

Examples:

$$\mathbf{A}^2 = \sum_{\alpha=1}^{3} \lambda_\alpha^2 \mathbf{n}_\alpha \otimes \mathbf{n}_\alpha, \quad \sqrt{\mathbf{A}} = \sum_{\alpha=1}^{3} \sqrt{\lambda_\alpha} \mathbf{n}_\alpha \otimes \mathbf{n}_\alpha, \quad \ln(\mathbf{A}) = \sum_{\alpha=1}^{3} \ln(\lambda_\alpha) \mathbf{n}_\alpha \otimes \mathbf{n}_\alpha$$

This spectral approach is widely used in constitutive models for hyperelastic materials.

3.14. Christoffel Symbols and Covariant Differentiation

3.14.1. Motivation: Why Naive Derivatives Fail in Curvilinear Coordinates

In Cartesian coordinates, the partial derivative of a vector is straightforward: only the **components** change, not the basis vectors. But in curvilinear coordinates, when we move from one point to a nearby point in the coordinate grid, the **basis vectors themselves change**. A naive partial derivative thus mixes changes in components with changes in basis, and the result is **not a tensor**.

Consider a vector field $\mathbf{u}(\theta^i) = u^j(\theta^i)\mathbf{g}_j(\theta^i)$. Taking the derivative:

$$\frac{\partial \mathbf{u}}{\partial \theta^i} = \frac{\partial u^j}{\partial \theta^i} \mathbf{g}_j + u^j \frac{\partial \mathbf{g}_j}{\partial \theta^i}$$

The second term accounts for basis change. The **Christoffel symbol** quantifies this change.

3.14.2. Christoffel Symbol (Symbol of the Second Kind)

The **Christoffel symbol of the second kind** Γ^k_{ij} is defined by:

$$\frac{\partial \mathbf{g}_j}{\partial \theta^i} = \Gamma^k_{ij} \mathbf{g}_k$$

Explicit formula (in terms of the metric tensor):

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{mj}}{\partial \theta^i} + \frac{\partial g_{im}}{\partial \theta^j} - \frac{\partial g_{ij}}{\partial \theta^m} \right)$$

Key property: The Christoffel symbol is **symmetric in the lower two indices**:

$$\Gamma^k_{ij} = \Gamma^k_{ji}$$

Important note: The Christoffel symbol is **not a tensor**. It vanishes in Cartesian coordinates (where basis vectors are constant), but is non-zero in general curvilinear systems.

Cylindrical coordinates (r, ϕ, z): Metric is $g_{rr} = 1$, $g_{\phi\phi} = r^2$, $g_{zz} = 1$, all others zero.

Non-zero Christoffel symbols:

$$\Gamma^r_{\phi\phi} = -r, \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \text{all others} = 0$$

Physical meaning: $\Gamma^r_{\phi\phi} = -r$ says that when you move in the ϕ direction twice, the radial basis vector shrinks, which makes sense geometrically (the coordinate grid circles get smaller as you go outward).

3.14.3. Covariant Derivative of a Vector

To obtain a **tensor derivative**, we define the **covariant derivative** of a contravariant vector u^i:

$$u^i_{;j} = \frac{\partial u^i}{\partial \theta^j} + \Gamma^i_{jk} u^k$$

Similarly, for a covariant vector u_i:

$$u_{i;j} = \frac{\partial u_i}{\partial \theta^j} - \Gamma^k_{ij} u_k$$

The semi-colon notation (;) denotes covariant differentiation. The result is a **tensor** (it transforms like a tensor under coordinate changes), unlike the naive partial derivative.

3.14.4. Covariant Derivative of a Tensor

For a general second-order tensor $\mathbf{A} = A^{kl} \mathbf{g}_k \otimes \mathbf{g}_l$:

$$A^{kl}_{;m} = \frac{\partial A^{kl}}{\partial \theta^m} + \Gamma^k_{mn} A^{nl} + \Gamma^l_{mn} A^{kn}$$

The pattern: **add** Christoffel-symbol corrections for each upper index, **subtract** for each lower index.

3.14.5. Divergence in Curvilinear Coordinates

A direct application: the **divergence** of a vector field is:

$$\nabla \cdot \mathbf{u} = u^i_{;i} = \frac{\partial u^i}{\partial \theta^i} + \Gamma^i_{ik} u^k = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \theta^i} \left(\sqrt{g}\, u^i \right)$$

where $g = \det g_{ij}$ is the metric determinant.

> **Note**
>
> **Physical interpretation:** Covariant derivatives encode the fact that in a curved or non-orthonormal coordinate system, "parallel transport" of a vector (moving it while keeping it fixed in the local frame) requires a compensation term. In finite-strain mechanics, material coordinates deform with the material, and these coordinate curves become non-orthogonal. The Christoffel symbols then represent the geometric effect of the deformation on how we measure spatial derivatives.

3.14.6. Tensor Character

The covariant derivatives are **components of tensors** under coordinate transformations. This is why Grad, Div, and Curl are valid tensor operations: they produce results that transform as tensors, making them coordinate-independent and physically meaningful.

3.15. Derivative of a Tensor with Respect to a Tensor

For $\mathbf{F} = \mathbf{F}(\mathbf{G})$ where both are second-order tensors:

$$\mathbb{H} = \frac{\partial \mathbf{F}}{\partial \mathbf{G}}, \quad H_{klij} = \frac{\partial F_{kl}}{\partial G_{ij}}$$

This is a **fourth-order tensor** with 81 components in 3D.

Useful identities:

$$\frac{\partial \mathbf{A}}{\partial \mathbf{A}} = \mathbb{I}, \quad \frac{\partial \mathrm{tr}(\mathbf{A})}{\partial \mathbf{A}} = \mathbf{I}, \quad \frac{\partial \det(\mathbf{A})}{\partial \mathbf{A}} = (\det \mathbf{A})\mathbf{A}^{-T}$$

These derivatives are essential when deriving stress-strain relationships in constitutive theory.

4. L02 — Tensor Calculus

Derivatives, Gradients, and Tensor-Valued Functions

4.1. Scalar Function of a Tensor

Let $f = f(\mathbf{F})$ be a scalar. Its derivative with respect to $\mathbf{F}$ is defined via the directional derivative:

$$df = \frac{\partial f}{\partial F_{ij}} dF_{ij} = \frac{\partial f}{\partial \mathbf{F}} : d\mathbf{F}$$

so $\frac{\partial f}{\partial \mathbf{F}}$ is a second-order tensor with components $\frac{\partial f}{\partial F_{ij}}$.

4.2. Tensor Function of a Tensor

The derivative of a second-order tensor $\mathbf{F}$ with respect to a second-order tensor $\mathbf{G}$ is a **fourth-order tensor**:

$$\mathbb{H} = \frac{\partial \mathbf{F}}{\partial \mathbf{G}}, \qquad H_{klij} = \frac{\partial F_{kl}}{\partial G_{ij}}.$$

Useful identity: derivative of a second-order tensor with respect to itself,

$$\frac{\partial \mathbf{A}}{\partial \mathbf{A}} = \mathbb{I}, \qquad \mathbb{I}_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

4.3. Derivatives with Respect to Invariants

For a symmetric tensor **A**, three principal invariants capture all geometric information (see A01 for notation):

$$I_1(\mathbf{A}) = \mathrm{tr}(\mathbf{A}), \quad I_2(\mathbf{A}) = \tfrac{1}{2}[\mathrm{tr}(\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)], \quad I_3(\mathbf{A}) = \det(\mathbf{A})$$

Many constitutive functions depend on **A** only through its invariants, so we must compute derivatives like $\partial f(I_1, I_2, I_3)/\partial \mathbf{A}$ by the chain rule:

$$\frac{\partial f}{\partial \mathbf{A}} = \frac{\partial f}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{A}} + \frac{\partial f}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{A}} + \frac{\partial f}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{A}}$$

Below we derive each invariant gradient from first principles.

4.3.1. Derivative of I_1 (Trace)

Starting with the definition $I_1(\mathbf{A}) = \mathrm{tr}(\mathbf{A}) = A_{ii}$, we use index notation:

$$\frac{\partial I_1}{\partial A_{mn}} = \frac{\partial}{\partial A_{mn}}(A_{ii}) = \delta_{im}\delta_{in} = \delta_{mn}$$

In tensor form, with the identity tensor **I**:

$$\boxed{\frac{\partial I_1}{\partial \mathbf{A}} = \mathbf{I}}$$

Physical meaning: The trace is linear in **A**, so its gradient is constant.

4.3.2. Derivative of I_2 (Second Invariant)

Starting with $I_2 = \frac{1}{2}(I_1^2 - \mathrm{tr}(\mathbf{A}^2))$, we differentiate each term:

$$\frac{\partial}{\partial A_{mn}} \frac{1}{2} I_1^2 = I_1 \frac{\partial I_1}{\partial A_{mn}} = I_1 \delta_{mn}$$

For the second term, $\mathrm{tr}(\mathbf{A}^2) = A_{ij} A_{ji}$:

$$\frac{\partial}{\partial A_{mn}} (A_{ij} A_{ji}) = \delta_{im} \delta_{jn} A_{ji} + A_{ij} \delta_{jm} \delta_{in} = A_{ni} + A_{im} = (A + A^T)_{mn}$$

Since $\mathbf{A}$ is symmetric, $\mathbf{A}^T = \mathbf{A}$:

$$\frac{\partial}{\partial A_{mn}} (A_{ij} A_{ji}) = 2A_{mn}$$

Combining:

$$\frac{\partial I_2}{\partial A_{mn}} = I_1 \delta_{mn} - A_{mn}$$

In tensor form:

$$\boxed{\frac{\partial I_2}{\partial \mathbf{A}} = I_1 \mathbf{I} - \mathbf{A}}$$

Physical meaning: This gradient involves both the trace and the tensor itself; it arises because I_2 mixes first-order and second-order contractions.

4.3.3. Derivative of I_3 (Determinant) — Full Derivation

This is the most subtle case. We use a perturbation argument. Let $\mathbf{A} + \epsilon\mathbf{B}$ be a perturbed tensor, and expand the determinant:

$$\det(\mathbf{A} + \epsilon\mathbf{B}) = \det\mathbf{A}\ \det(\mathbf{I} + \epsilon\mathbf{A}^{-1}\mathbf{B})$$

For small ϵ, the characteristic polynomial of $\epsilon\mathbf{A}^{-1}\mathbf{B}$ gives:

$$\det(\mathbf{I} + \epsilon\mathbf{A}^{-1}\mathbf{B}) = 1 + \epsilon\,\mathrm{tr}(\mathbf{A}^{-1}\mathbf{B}) + O(\epsilon^2)$$

Substituting back:

$$\det(\mathbf{A} + \epsilon\mathbf{B}) = \det\mathbf{A}\left(1 + \epsilon\,\mathrm{tr}(\mathbf{A}^{-1}\mathbf{B}) + O(\epsilon^2)\right)$$

Now, $\mathrm{tr}(\mathbf{A}^{-1}\mathbf{B}) = (\mathbf{A}^{-T})_{ij}B_{ij}$, so the Fréchet derivative is:

$$\delta(\det\mathbf{A}) = \det\mathbf{A}\,(\mathbf{A}^{-T})_{ij}\delta A_{ij}$$

By the chain rule definition of the gradient:

$$\boxed{\frac{\partial I_3}{\partial\mathbf{A}} = \frac{\partial(\det\mathbf{A})}{\partial\mathbf{A}} = I_3\mathbf{A}^{-T} = (\det\mathbf{A})\mathbf{A}^{-T}}$$

Physical meaning: The derivative couples the determinant (volume change) with the inverse-transpose of the tensor. This makes sense dimensionally: to measure how volume changes with strain, we need the strain-inverse.

4.3.4. Summary Box

Invariant Derivatives (Quick Reference)

For a symmetric tensor $\mathbf{A}$ with invariants I_1, I_2, I_3:

$$\frac{\partial I_1}{\partial \mathbf{A}} = \mathbf{I}$$
$$\frac{\partial I_2}{\partial \mathbf{A}} = I_1\mathbf{I} - \mathbf{A}$$
$$\frac{\partial I_3}{\partial \mathbf{A}} = I_3\mathbf{A}^{-T}$$

Use these with the chain rule to compute derivatives of any invariant-based function.

4.3.5. Worked Example: Derivative of $\mathrm{tr}(\mathbf{A}^2)$

Suppose we want $\partial(\mathrm{tr}(\mathbf{A}^2))/\partial\mathbf{A}$. Note that $\mathrm{tr}(\mathbf{A}^2) = I_1^2 - 2I_2$.

By the chain rule:

$$\frac{\partial}{\partial \mathbf{A}}(I_1^2 - 2I_2) = 2I_1\frac{\partial I_1}{\partial \mathbf{A}} - 2\frac{\partial I_2}{\partial \mathbf{A}}$$

$$= 2I_1\mathbf{I} - 2(I_1\mathbf{I} - \mathbf{A}) = 2I_1\mathbf{I} - 2I_1\mathbf{I} + 2\mathbf{A} = 2\mathbf{A}$$

This makes sense: $\mathrm{tr}(\mathbf{A}^2)$ is bilinear in $\mathbf{A}$, so the gradient scales linearly with $\mathbf{A}$.

4.4. The Gradient Operator

For a scalar field $\phi(\mathbf{x})$:

$$\nabla\phi = \frac{\partial \phi}{\partial x_i}\mathbf{e}_i$$

For a vector field $\mathbf{v}(\mathbf{x})$:

$$(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j} \quad \Rightarrow \quad \text{a second-order tensor}$$

Divergence of a vector: $\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i}$

Divergence of a second-order tensor: $(\nabla \cdot \mathbf{T})_i = \frac{\partial T_{ij}}{\partial x_j}$

4.5. Integral Theorems

Gauss (Divergence) Theorem:

$$\int_{\Omega} \nabla \cdot \mathbf{v} \, dV = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, dA$$

For a tensor:

$$\int_{\Omega} \nabla \cdot \mathbf{T} \, dV = \int_{\partial\Omega} \mathbf{T}\mathbf{n} \, dA$$

These theorems convert body integrals to boundary integrals — essential for the weak form in FEM. We will expand on these theorems and their applications in Section "Gauss Divergence Theorem" later in this chapter.

4.6. Objective Rates

4.6.1. What is Objectivity?

A fundamental principle of mechanics states that **material properties are independent of the observer**. A quantity is called **objective** (or **frame-indifferent**) if its form does not change when observed from different (moving) reference frames related by a superposed rigid-body motion.

Consider two observers: one in a fixed frame and one in a frame undergoing a superposed rigid-body motion with orthogonal rotation tensor $\mathbf{Q}(t)$ and translation $\mathbf{c}(t)$:

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t), \quad \mathbf{Q}^T(t)\mathbf{Q}(t) = \mathbf{I}, \quad \det \mathbf{Q} = 1$$

Different tensor quantities transform differently:

- **Objective scalar:** $\phi^* = \phi$ (unchanged)
- **Objective vector:** $\mathbf{v}^* = \mathbf{Q}\mathbf{v}$ (rotates with frame)
- **Objective 2nd-order tensor:** $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$ (rotates with frame on both legs)
- **Two-point tensor (non-objective):** $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$ (rotates on left leg only)

4.6.2. Objectivity of Kinematic Tensors

Right Cauchy-Green tensor: Since $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$:

$$\mathbf{C}^* = (\mathbf{F}^*)^T\mathbf{F}^* = \mathbf{F}^T\mathbf{Q}^T\mathbf{Q}\mathbf{F} = \mathbf{F}^T\mathbf{F} = \mathbf{C}$$

So $\mathbf{C}$ is **invariant** (objective in the reference configuration).

Green-Lagrange strain: Similarly, $\mathbf{E}^* = \frac{1}{2}(\mathbf{C}^* - \mathbf{I}) = \mathbf{E}$. **Invariant.**

Left Cauchy-Green (Finger) tensor: In the current configuration:

$$\mathbf{b}^* = \mathbf{F}^*(\mathbf{F}^*)^T = \mathbf{Q}\mathbf{F}\mathbf{F}^T\mathbf{Q}^T = \mathbf{Q}\mathbf{b}\mathbf{Q}^T$$

So $\mathbf{b}$ is **objective** (transforms as a proper 2nd-order tensor).

Almansi strain: Similarly, $\mathbf{e}^* = \mathbf{Q}\mathbf{e}\mathbf{Q}^T$. **Objective.**

4.6.3. Objectivity of Stress Tensors

Cauchy stress (force per current area) transforms as:

$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T$$

Objective.

1st Piola-Kirchhoff stress (force per reference area):

$$\mathbf{P}^* = \mathbf{Q}\mathbf{P}$$

Two-point tensor (not fully objective; the right leg does not rotate because area is in the reference frame).

2nd Piola-Kirchhoff stress (purely material):

$$\mathbf{S}^* = \mathbf{S}$$

Invariant (objective scalar-like in the reference frame).

4.6.4. The Problem with Material Time Derivatives

An objective tensor $\mathbf{T}$ remains objective under the superposed motion, but its material time derivative does **not**:

$$\dot{\mathbf{T}}^* = \frac{d}{dt}[\mathbf{Q}(t)\mathbf{T}(t)\mathbf{Q}^T(t)]$$

Using the product rule:

$$\dot{\mathbf{T}}^* = \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^T$$

If $\dot{\mathbf{T}}^* = \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T$ were to hold, the first and third terms would vanish. But they don't—they involve $\dot{\mathbf{Q}}$.

Define the **spin of the superposed frame:**

$$\boldsymbol{\Omega} = \dot{\mathbf{Q}}^T \mathbf{Q}$$

(skew-symmetric). Then:

$$\dot{\mathbf{T}}^* = \mathbf{Q}(\dot{\mathbf{T}} - \boldsymbol{\Omega}\mathbf{T} + \mathbf{T}\boldsymbol{\Omega}^T)\mathbf{Q}^T$$

The material time derivative $\dot{\mathbf{T}}$ does not transform as an objective tensor.

4.6.5. Consequences for Constitutive Equations

Consider a hypoelastic law $\dot{\boldsymbol{\sigma}} = \mathbb{H} : \mathbf{d}$. If the frame rotates ($\boldsymbol{\Omega} \neq 0$) but the stress does not change in the rotating frame ($\dot{\boldsymbol{\sigma}}^* = 0$), then:

$$\dot{\boldsymbol{\sigma}} = \mathbf{Q}(-\boldsymbol{\Omega}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\Omega}^T)\mathbf{Q}^T \neq 0$$

This would induce spurious strains $\mathbf{d} \neq 0$ even though the material state is unchanged in the rotating frame—a **physically inadmissible result**.

Solution: Use objective rate tensors that automatically subtract off the spin contributions.

4.6.6. Jaumann (Corotational) Rate

The **Jaumann rate** removes the spin of the continuum itself. Let $\mathbf{W} = \text{skew}(\mathbf{l})$ be the continuum spin tensor (with axial vector $\mathbf{w}$ such that $\mathbf{W}\mathbf{v} = \mathbf{w} \times \mathbf{v}$):

$$\boxed{\overset{\triangledown}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}}$$

Proof of objectivity:

$$\overset{\triangledown}{\mathbf{T}}^* = \dot{\mathbf{T}}^* - \mathbf{W}^*\mathbf{T}^* + \mathbf{T}^*(\mathbf{W}^*)$$

where $\mathbf{W}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^T$ (since $\mathbf{l}^* = \mathbf{Q}(\mathbf{l} - \boldsymbol{\Omega})\mathbf{Q}^T$, the skew part gives $\mathbf{W}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^T$). Substituting:

$$\overset{\triangledown}{\mathbf{T}}^* = \mathbf{Q}(\dot{\mathbf{T}} - \boldsymbol{\Omega}\mathbf{T} + \mathbf{T}\boldsymbol{\Omega}^T)\mathbf{Q}^T - \mathbf{Q}\mathbf{W}\mathbf{Q}^T\mathbf{Q}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\mathbf{Q}^T\mathbf{Q}\mathbf{W}\mathbf{Q}^T$$

$$= \mathbf{Q}[\dot{\mathbf{T}} - \boldsymbol{\Omega}\mathbf{T} + \mathbf{T}\boldsymbol{\Omega}^T - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}]\mathbf{Q}^T$$

The $\boldsymbol{\Omega}$ terms cancel if $\boldsymbol{\Omega}$ is skew (which it is), leaving:

$$\overset{\triangledown}{\mathbf{T}}^* = \mathbf{Q}[\dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W}]\mathbf{Q}^T = \mathbf{Q}\overset{\triangledown}{\mathbf{T}}\mathbf{Q}^T$$

Objective!

Physical interpretation: The Jaumann rate removes the rigid rotation of the continuum. In a frame that co-rotates with the material, the rate is computed in that rotating frame.

4.6.7. Oldroyd (Upper-Convected) Rate

The **Oldroyd rate** uses the full velocity gradient (not just spin):

$$\boxed{\overset{\triangle}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T}$$

where $\mathbf{L} = \mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the velocity gradient.

Proof (sketch): Under the superposed motion, $\mathbf{L}^* = \mathbf{Q}(\mathbf{L} - \boldsymbol{\Omega})\mathbf{Q}^T$, so:

$$\overset{\triangle}{\mathbf{T}}^* = \mathbf{Q}(\dot{\mathbf{T}} - \boldsymbol{\Omega}\mathbf{T} + \mathbf{T}\boldsymbol{\Omega}^T - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T)\mathbf{Q}^T$$

Again the $\boldsymbol{\Omega}$ spin terms cancel (skew-symmetric), and:

$$\overset{\triangle}{\mathbf{T}}^* = \mathbf{Q}\overset{\triangle}{\mathbf{T}}\mathbf{Q}^T$$

Objective!

Physical interpretation: The Oldroyd rate accounts for the full deformation (strain + rotation) in the current configuration. It is the Lie derivative along the velocity field, and is commonly used in viscoelasticity and finite-strain plasticity.

4.6.8. Truesdell Rate

The **Truesdell rate** additionally corrects for volume change:

$$\boxed{\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{L}\mathbf{T} - \mathbf{T}\mathbf{L}^T + (\operatorname{tr}\mathbf{L})\mathbf{T}}$$

The term $(\operatorname{tr}\mathbf{L})\mathbf{T}$ (proportional to the volumetric strain rate $\operatorname{tr}\mathbf{d}$) restores objective status when volume changes couple to stress evolution.

Objectivity proof: Similar to Oldroyd, with $\operatorname{tr}\mathbf{L}$ being objective.

Physical interpretation: Used in hyperelastic models where the strain energy couples to both distortion and volume. The Truesdell rate ensures that hydrostatic stress changes are handled correctly.

4.6.9. Summary and Comparison

Objective Rate Tensors

Rate	Formula	Use Case
Jaumann	$\overset{\triangledown}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{WT} + \mathbf{TW}$	Small-strain plasticity, materials with negligible convection
Oldroyd	$\overset{\triangle}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{LT} - \mathbf{TL}^T$	Viscoelasticity, finite-strain models with convection
Truesdell	$\overset{\circ}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{LT} - \mathbf{TL}^T + (\operatorname{tr}\mathbf{L})\mathbf{T}$	Hyperelasticity with volumetric coupling

All three **transform as objective 2nd-order tensors** under superposed rigid motions, making them suitable for frame-indifferent constitutive laws.

4.6.10. Worked Example: Jaumann Rate in Simple Shear

Consider a material element undergoing simple shear at constant shear rate $\dot{\gamma}$:

$$\mathbf{v} = \dot{\gamma} y \mathbf{e}_1, \quad \mathbf{L} = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The spin and strain rate are:

$$\mathbf{W} = \frac{1}{2}\begin{pmatrix} 0 & \dot{\gamma} & 0 \\ -\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{D} = \frac{1}{2}\begin{pmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For a stress state $\boldsymbol{\sigma} = \sigma_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + \sigma_{12}\mathbf{e}_2 \otimes \mathbf{e}_1$ (shear only), the material time derivative is $\dot{\boldsymbol{\sigma}} = \dot{\sigma}_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$.

The Jaumann rate modifies this by:

$$\overset{\triangledown}{\boldsymbol{\sigma}}_{12} = \dot{\sigma}_{12} - \mathbf{W}\boldsymbol{\sigma} \cdot \mathbf{e}_2 \otimes \mathbf{e}_1 + \cdots$$

This removes the spurious stress oscillations that would arise from the material element spinning without actual shear strain changing—a well-known issue in nonlinear dynamics simulations.

4.7. Push-Forward and Pull-Back

Mapping between material (Lagrangian) and spatial (Eulerian) descriptions:

Operation	Formula
Push-forward of covariant 2-tensor	$\varphi_* \mathbf{S} = \mathbf{F}^{-T}\mathbf{S}\mathbf{F}^{-1}$
Pull-back of contravariant 2-tensor	$\varphi^* \boldsymbol{\tau} = \mathbf{F}^T \boldsymbol{\tau} \mathbf{F}$
Lie derivative	$\mathfrak{L}_v \mathbf{b} = \mathbf{F}\, \frac{d}{dt}(\mathbf{F}^{-1}\mathbf{b}\mathbf{F}^{-T})\, \mathbf{F}^T$

4.8. Deformation Gradient and Kinematics

Consider a body deforming between: - **Reference configuration** $\otimes_0$ with coordinates $\mathbf{X}$ - **Current configuration** $\otimes_t$ with coordinates $\mathbf{x}$

The mapping: $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$

The **deformation gradient:**

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \nabla_X \mathbf{x}$$

is a two-point tensor relating infinitesimal material line elements:

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$$

Key property: $J = \det(\mathbf{F}) > 0$ (volume ratio between configurations).

4.9. Right and Left Cauchy-Green Tensors

The **right Cauchy-Green deformation tensor:**

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$$

Symmetric, positive-definite, describes deformation in the reference configuration (material description).

The **left Cauchy-Green (Finger) deformation tensor:**

$$\mathbf{b} = \mathbf{F} \cdot \mathbf{F}^T$$

Symmetric, positive-definite, describes deformation in the current configuration (spatial description). **Lowercase** per the canonical convention (A.3.1, A.4): $\mathbf{b}$ lives on the current configuration, while the uppercase $\mathbf{C}$ lives on the reference configuration.

Relationship: $\mathbf{b} = \mathbf{F} \cdot \mathbf{C} \cdot \mathbf{F}^{-T}$ (spatial push-forward of $\mathbf{C}$).

4.10. Strain Tensors

Green-Lagrange strain tensor (material description):

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I})$$

Almansi strain tensor (spatial description):

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{b}^{-1}) = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-1} \cdot \mathbf{F}^{-T})$$

Relationship: Pull-back of Almansi to reference config gives Green-Lagrange:

$$\mathbf{E} = \mathbf{F}^T \cdot \mathbf{e} \cdot \mathbf{F}$$

Small-strain limit: For small displacements $\mathbf{u}$ with $|\nabla \mathbf{u}| \ll 1$:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) = \mathrm{sym}(\nabla \mathbf{u})$$

4.11. Stretch and Rotation: Polar Decomposition

The deformation gradient admits a unique decomposition:

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$$

where: - $\mathbf{R}$: orthogonal rotation tensor ($\mathbf{R}^T\mathbf{R} = \mathbf{I}$, $\det(\mathbf{R}) = 1$) - $\mathbf{U}$: right stretch tensor (symmetric, positive-definite, in reference config) - $\mathbf{V}$: left stretch tensor (symmetric, positive-definite, in current config)

Relations:

$$\mathbf{U} = \sqrt{\mathbf{C}}, \quad \mathbf{V} = \sqrt{\mathbf{b}}, \quad \mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$$

Spectral decomposition: $\mathbf{C}$ and $\mathbf{b}$ share eigenvalues λ_i^2 (principal stretches squared), but eigenvectors differ — $\mathbf{N}_i$ (reference, uppercase) for $\mathbf{C}$, $\mathbf{n}_i$ (current, lowercase) for $\mathbf{b}$:

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{N}_i \otimes \mathbf{N}_i, \quad \mathbf{b} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{n}_i \otimes \mathbf{n}_i$$

with $\mathbf{n}_i = \mathbf{R}\mathbf{N}_i$.

4.12. Volume and Area Changes

Volume change: A volume element transforms as

$$dV = J\,dV_0, \quad J = \det(\mathbf{F})$$

Isochoric/isovolumetric decomposition:

$$\mathbf{F} = J^{1/3}\bar{\mathbf{F}}, \quad \det(\bar{\mathbf{F}}) = 1$$

where $\bar{\mathbf{F}}$ represents the shape-changing (distortional) part and $J^{1/3}$ represents the volumetric part.

Area change (Nanson's formula): Surface normal transforms as

$$\mathbf{n}\,dA = J\mathbf{F}^{-T} \cdot \mathbf{N}\,dA_0$$

where $\mathbf{N}$ and $\mathbf{n}$ are normals in reference and current configurations.

4.13. Velocity Gradient and Rate Tensors

Velocity gradient:

$$\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}$$

where $\mathbf{v} = \dot{\mathbf{x}}$ is the material velocity.

Decomposition into symmetric and skew-symmetric parts:

$$\mathbf{l} = \mathbf{d} + \mathbf{w}$$

- **Strain rate tensor:** $\mathbf{d} = \mathrm{sym}(\mathbf{l}) = \frac{1}{2}(\mathbf{l} + \mathbf{l}^T)$ (objective)
- **Spin tensor:** $\mathbf{w} = \mathrm{skew}(\mathbf{l}) = \frac{1}{2}(\mathbf{l} - \mathbf{l}^T)$ (not objective)

Volumetric strain rate:

$$\dot{J} = J\,\mathrm{tr}(\mathbf{d}) = J\,\mathrm{div}(\mathbf{v})$$

4.14. Objective Stress Measures

Cauchy (true) stress $\boldsymbol{\sigma}$: Force per current area.

$$\text{Transformation: } \boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T \quad \text{(objective)}$$

Nominal stress (1st Piola-Kirchhoff) $\mathbf{P}$: Force per reference area (one-point tensor).

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}, \quad \text{Transformation: } \mathbf{P}^* = \mathbf{Q}\mathbf{P} \quad \text{(two-point, not objective)}$$

2nd Piola-Kirchhoff stress $\mathbf{S}$: Conjugate to Green strain (material description).

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}, \quad \text{Transformation: } \mathbf{S}^* = \mathbf{S} \quad \text{(invariant/objective)}$$

Kirchhoff stress:

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = \mathbf{F}\mathbf{S}\mathbf{F}^T$$

4.15. Objectivity and Frame-Indifference

As defined in Section "Objective Rates," a spatial tensor $\mathbf{T}$ is **objective** (frame-indifferent) if, under a superimposed rigid motion $\mathbf{Q}(t)$:

$$\mathbf{T}^* = \mathbf{Q}(t)\mathbf{T}\mathbf{Q}^T(t)$$

Problem with material time derivative: The rate $\dot{\mathbf{T}}$ of an objective tensor is **not** objective:

$$\dot{\mathbf{T}}^* \neq \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T$$

Instead:

$$\dot{\mathbf{T}}^* = \mathbf{Q}(\dot{\mathbf{T}} - \mathbf{\Omega}\mathbf{T} + \mathbf{T}\mathbf{\Omega}^T)\mathbf{Q}^T$$

where $\mathbf{\Omega} = \dot{\mathbf{Q}}^T\mathbf{Q}$ is the spin of the rotating frame.

4.16. Gauss Divergence Theorem

Building on the integral theorems introduced earlier (Section "Integral Theorems"), we now explore their applications in depth. For a region Ω with boundary $\partial\Omega$ and outward normal $\mathbf{n}$:

Vector form:

$$\int_\Omega \nabla \cdot \mathbf{v}\, dV = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, dA$$

Tensor form:

$$\int_\Omega \nabla \cdot \mathbf{T}\, dV = \int_{\partial\Omega} \mathbf{T}\mathbf{n}\, dA$$

Essential for: - Deriving equilibrium equations from stress divergence - Formulating weak forms in finite element methods - Relating volume and surface integrals in conservation laws

Example: From $\int_\Omega (\nabla \cdot \boldsymbol{\sigma})\, dV = -\rho \int_\Omega \mathbf{a}\, dV$ (equilibrium without body force), we get:

$$\int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n}\, dA = -\rho \int_\Omega \mathbf{a}\, dV$$

5. L03 — Continuum Kinematics

Deformation, Strain Measures, and Stress Tensors

> **Sign convention.** The Cauchy stress $\boldsymbol{\sigma}$ is **positive in tension** and **negative in compression** throughout this course. The hydrostatic pressure is $p = -\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}$ (so positive p means compression).

> **Notation from this chapter onwards.** Direct tensor notation (e.g. $\mathbf{F}$, $\boldsymbol{\sigma}$, $\mathbb{C}^e$) is used predominantly; index notation (F_{iJ}, σ_{ij}, C^e_{ijkl}) is shown where clarity requires it. Bold Latin uppercase denotes reference-configuration second-order tensors ($\mathbf{F}$, $\mathbf{C}$, $\mathbf{E}$); bold Latin lowercase denotes current-configuration second-order tensors ($\mathbf{b}$, $\mathbf{e}$); blackboard bold ($\mathbb{C}$, $\mathbb{P}$) denotes fourth-order tensors. See L02 for the index-notation toolkit.

5.1. The Deformation Map

A body $\mathcal{B}$ occupies a **reference (material, Lagrangian) configuration** Ω_0 at some initial time. After deformation, it occupies a **current (spatial, Eulerian) configuration** Ω_t at time t.

The **deformation map** (or **motion**) $\boldsymbol{\varphi}$ is a smooth bijection that carries each material point from the reference configuration to the current configuration:

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t), \qquad \boldsymbol{\varphi} : \Omega_0 \times [0, T] \to \Omega_t.$$

The **displacement field** is defined as:

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x} - \mathbf{X} = \boldsymbol{\varphi}(\mathbf{X}, t) - \mathbf{X}.$$

By convention (notation canon, A01): - **Reference-config quantities** use uppercase Latin indices or letters: $\mathbf{X}$ (position), dV, dA (volume/area elements), $\mathbf{N}$ (normal), ρ_0 (density). - **Current-config quantities** use lowercase: $\mathbf{x}, dv, da, \mathbf{n}, \rho$.

5.2. Material vs. Spatial Descriptions

A field quantity ϕ can be expressed in two ways:

Material (Lagrangian) description: $\phi(\mathbf{X}, t)$ — follow a specific material point as it moves.

Spatial (Eulerian) description: $\phi(\mathbf{x}, t)$ — observe the field at fixed spatial locations.

These are related by the deformation map. For example, if a scalar field has material form $\phi_m(\mathbf{X}, t)$, its spatial form is:

$$\phi_s(\mathbf{x}, t) = \phi_m(\boldsymbol{\varphi}^{-1}(\mathbf{x}, t), t).$$

The material time derivative (following a material point) is:

$$\dot{\phi} = \frac{D\phi}{Dt} = \left.\frac{\partial \phi_s}{\partial t}\right|_{\mathbf{x}} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \phi_s = \frac{\partial \phi_s}{\partial t} + v_i \frac{\partial \phi_s}{\partial x_i},$$

where $\mathbf{v}(\mathbf{x}, t) = \dot{\boldsymbol{\varphi}}(\mathbf{X}, t)$ is the velocity field. The first term is the **local rate** (change at a fixed point); the second is the **advective rate** (material sweeping through the field).

Physical interpretation: In a laboratory experiment, we may measure quantities at fixed locations (Eulerian, e.g., temperature at a probe). In a material simulation, we track each grain of material (Lagrangian, e.g., stress at a Gauss point). Both views are valid; we switch between them using $\mathbf{v}$ and $\mathbf{F}$.

	Material	Spatial
Symbol	$\phi(\mathbf{X}, t)$	$\phi(\mathbf{x}, t)$
Meaning	Follow material point	Observe at fixed location
Time derivative	$\partial\phi/\partial t\|_{\mathbf{X}}$	$D\phi/Dt = \partial\phi/\partial t\|_{\mathbf{x}} + \mathbf{v} \cdot \nabla_{\mathbf{x}}\phi$
Example	$T(\mathbf{X}, t)$ in a grain	$T(\mathbf{x}, t)$ at a thermometer location

5.3. The Deformation Gradient — Extensive Treatment

5.3.1. Definition and Basic Properties

The **deformation gradient F** is the Jacobian matrix of the deformation map:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \nabla_{\mathbf{X}} \boldsymbol{\varphi}.$$

In index notation (with lowercase i for current config, uppercase J for reference):

$$F_{iJ} = \frac{\partial x_i}{\partial X_J}.$$

Two-point tensor character: This is the defining feature of $\mathbf{F}$. The first index lives in the current configuration (lowercase = spatial); the second in the reference configuration (uppercase = material). As a result, $\mathbf{F}$ is **not purely a spatial or purely a material tensor**. It maps vectors from material space to spatial space:

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X}.$$

This duality is crucial: $\mathbf{F}$ lives in both configurations simultaneously and measures the local stretching and rotation caused by $\boldsymbol{\varphi}$.

The Jacobian determinant:

$$J = \det \mathbf{F} > 0$$

must be strictly positive for a physical (orientation-preserving, non-interpenetrating) deformation. J is the local volume ratio:

$$dv = J\, dV.$$

Physical interpretation: If we zoom in on an infinitesimal material volume element around a point, the deformation gradient tells us how that element's shape and size change. A rigid-body rotation has $\mathbf{F} = \mathbf{R}$ (orthogonal, $J = 1$). Stretching changes the eigenvalues of $\mathbf{F}$. Compression reduces J below 1; expansion increases it above 1.

5.3.2. Geometric Mappings Induced by $\mathbf{F}$

Line elements: An infinitesimal line segment $d\mathbf{X}$ in the reference config maps to:

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X}.$$

Area elements (Nanson's formula): A surface element with current normal $\mathbf{n}$ and area da relates to the reference element (normal $\mathbf{N}$, area dA) by:

$$\mathbf{n}\,da = J\mathbf{F}^{-T}\mathbf{N}\,dA.$$

Derivation sketch: The oriented area element is a vector perpendicular to the surface. Infinitesimal parallelograms in the reference config with sides $d\mathbf{X}_1, d\mathbf{X}_2$ and normal $\mathbf{N}\,dA$ map to spatial parallelograms with sides $\mathbf{F}\,d\mathbf{X}_1, \mathbf{F}\,d\mathbf{X}_2$. Their normal is proportional to $(\mathbf{F}\,d\mathbf{X}_1) \times (\mathbf{F}\,d\mathbf{X}_2) = J\mathbf{F}^{-T}(d\mathbf{X}_1 \times d\mathbf{X}_2)$ (by the adjugate formula). Thus $\mathbf{n}\,da = J\mathbf{F}^{-T}\mathbf{N}\,dA$.

Volume elements:

$$dv = J\,dV.$$

Derivation: An infinitesimal parallelepiped with edges $d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3$ has volume $dV = |d\mathbf{X}_1 \cdot (d\mathbf{X}_2 \times d\mathbf{X}_3)|$. Mapping to the current config: $dv = |(\mathbf{F}\,d\mathbf{X}_1) \cdot ((\mathbf{F}\,d\mathbf{X}_2) \times (\mathbf{F}\,d\mathbf{X}_3))| = |\det \mathbf{F}|\,dV = J\,dV$.

5.3.3. Push-Forward and Pull-Back

Given $\mathbf{F}$, we can move quantities between configurations:

Push-forward of vectors: $\mathbf{v}$ (ref) $\to$ $\mathbf{v}$ (curr): $\mathbf{v}_{\text{curr}} = \mathbf{F}\mathbf{v}_{\text{ref}}$.

Pull-back of vectors: $\mathbf{v}$ (curr) $\to$ $\mathbf{v}$ (ref): $\mathbf{v}_{\text{ref}} = \mathbf{F}^{-1}\mathbf{v}_{\text{curr}}$.

Push-forward of tensors: $\mathbf{A}$ (ref, second-order) $\to$ $\mathbf{A}$ (curr): $\mathbf{A}_{\text{curr}} = \mathbf{F}\mathbf{A}_{\text{ref}}\mathbf{F}^T$.

Pull-back of tensors: $\mathbf{A}$ (curr) $\to$ $\mathbf{A}$ (ref): $\mathbf{A}_{\text{ref}} = \mathbf{F}^{-T}\mathbf{A}_{\text{curr}}\mathbf{F}^{-1}$.

(These definitions ensure that inner products are preserved: if $\mathbf{u}_{\text{ref}} \to \mathbf{v}_{\text{curr}} = \mathbf{F}\mathbf{u}_{\text{ref}}$ and $\mathbf{v}_{\text{ref}} \to \mathbf{w}_{\text{curr}} = \mathbf{F}\mathbf{v}_{\text{ref}}$, then $\mathbf{v}_{\text{curr}} \cdot \mathbf{w}_{\text{curr}} = (\mathbf{F}\mathbf{u})^T(\mathbf{F}\mathbf{v}) = \mathbf{u}^T(\mathbf{F}^T\mathbf{F})\mathbf{v}$ involves the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.)

5.3.4. Worked Example: Simple Shear

Consider the deformation $x_1 = X_1 + \gamma X_2, x_2 = X_2, x_3 = X_3$ (shear in the x_1-x_2 plane with shear strain γ).

$$\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J = \det \mathbf{F} = 1.$$

The deformation is isochoric (volume-preserving). A line element initially along $\mathbf{e}_2$ (the 1,0,0]^T direction in the reference) maps to $(1, \gamma, 0)^T$ in the current — it rotates and stretches due to the shear.

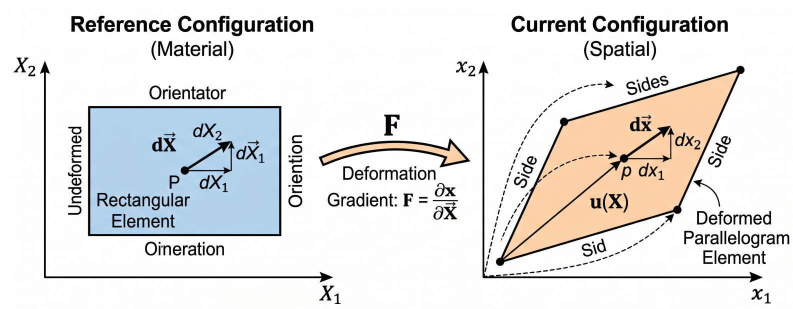


Figure 5.1.: The deformation map $\boldsymbol{\chi}$ carries a material point $\mathbf{X}$ in the reference configuration Ω_0 to its image $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ in the current configuration Ω. The deformation gradient $\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X}$ is the local linearisation of this map: it carries the infinitesimal line element $d\mathbf{X}$ at $\mathbf{X}$ to $d\mathbf{x} = \mathbf{F}\, d\mathbf{X}$ at $\mathbf{x}$. Because $\mathbf{F}$ has one leg in each configuration (reference-index J, current-index i), it is a **two-point tensor**.

5.3.5. Worked Example: Uniaxial Stretch

A uniform stretch: $x_1 = \lambda_1 X_1, x_2 = \lambda_2 X_2, x_3 = \lambda_3 X_3$.

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad J = \lambda_1 \lambda_2 \lambda_3.$$

The principal stretches are the diagonal entries. If $\lambda_1 > 1$ and $\lambda_2 = \lambda_3 < 1$ with $\lambda_1 \lambda_2^2 = 1$, the material stretches in one direction and compresses perpendicular to preserve volume (like squeezing a rubber ball in one axis).

5.3.6. Worked Example: Rigid-Body Rotation

A rotation about the $\mathbf{e}_3$ axis by angle θ:

$$\mathbf{F} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{R}.$$

Here, $\mathbf{F} = \mathbf{R}$ (orthogonal), $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, and $J = 1$. The right Cauchy-Green tensor is $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{R}^T\mathbf{R} = \mathbf{I}$. **No strain occurs** (shape and volume are unchanged); the material simply rotates as a rigid body.

5.4. Strain Measures — Extensive Treatment

5.4.1. Motivation: Measuring Change in Length

Consider two neighboring material points separated by an infinitesimal line element $d\mathbf{X}$ in the reference config. After deformation, they are separated by $d\mathbf{x} = \mathbf{F}\, d\mathbf{X}$.

The squared lengths are:

$$dS^2 = d\mathbf{X} \cdot d\mathbf{X}, \quad ds^2 = d\mathbf{x} \cdot d\mathbf{x} = (d\mathbf{X})^T \mathbf{F}^T \mathbf{F}\, d\mathbf{X}.$$

We seek a **strain measure** that: 1. Vanishes when there is no deformation ($d\mathbf{x} = d\mathbf{X}$). 2. Is zero for rigid rotations (where shape is unchanged). 3. Captures the change in length and angle between material line elements.

The key insight is that:

$$ds^2 - dS^2 = d\mathbf{X}^T (\mathbf{F}^T \mathbf{F} - \mathbf{I})\, d\mathbf{X} = 2\, d\mathbf{X}^T \mathbf{E}\, d\mathbf{X},$$

where $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})$ is the **Green-Lagrange strain tensor**.

5.4.2. Right Cauchy-Green Deformation Tensor

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}.$$

In index notation: $C_{JK} = F_{iJ} F_{iK}$ (material indices).

Properties: - Symmetric and positive-definite. - Lives in the reference configuration (both indices uppercase). - $\mathbf{C} = \mathbf{I}$ when there is no deformation.

Physical interpretation: $\mathbf{C}$ encodes all the deformation information. If we compute $d\mathbf{X}^T \mathbf{C}\, d\mathbf{X}$, we get ds^2, the squared length in the current config. The components of $\mathbf{C}$ in a given direction tell us how much that direction has been stretched and rotated.

5.4.3. Green-Lagrange Strain Tensor

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{I}) = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}).$$

Derivation: From $ds^2 - dS^2 = d\mathbf{X}^T(\mathbf{C} - \mathbf{I})\, d\mathbf{X}$, we define $\mathbf{E}$ so that:

$$ds^2 - dS^2 = 2\, d\mathbf{X}^T \mathbf{E}\, d\mathbf{X}.$$

The factor of 2 is conventional; it makes the strain "compatible" with the infinitesimal strain (see below).

Properties: - Symmetric, material-configuration tensor. - $\mathbf{E} = \mathbf{0}$ means $\mathbf{C} = \mathbf{I}$, i.e., no deformation. - For a rigid rotation, $\mathbf{F} = \mathbf{R}$, so $\mathbf{E} = \frac{1}{2}(\mathbf{R}^T\mathbf{R} - \mathbf{I}) = \mathbf{0}$ ✓

In terms of displacement: Since $\mathbf{F} = \mathbf{I} + \nabla_{\mathbf{X}}\mathbf{u}$,

$$\mathbf{E} = \tfrac{1}{2}(\nabla_{\mathbf{X}}\mathbf{u} + \nabla_{\mathbf{X}}\mathbf{u}^T + (\nabla_{\mathbf{X}}\mathbf{u})^T\nabla_{\mathbf{X}}\mathbf{u}).$$

This is the **finite-strain displacement-strain relation**. The first two terms (linear in $\nabla_{\mathbf{X}}\mathbf{u}$) are the **infinitesimal part**; the last (quadratic) is the **geometric nonlinearity**.

5.4.4. Left Cauchy-Green / Finger Deformation Tensor

$$\mathbf{b} = \mathbf{F}\mathbf{F}^T.$$

In index notation: $b_{ij} = F_{iJ}F_{jJ}$ (spatial indices).

Properties: - Symmetric and positive-definite, spatial-configuration tensor. - $\mathbf{b} = \mathbf{I}$ when there is no deformation.

Relation to C: They are not equal, but both encode the stretch information through their eigenvalues. We have:

$$(\mathbf{F}^T\mathbf{F})(\mathbf{F}^{-1}\mathbf{F}^{-T}) = \mathbf{F}^T\mathbf{F},$$

which is different. However, the right and left stretches (to be introduced below) have the same eigenvalues, and so do $\mathbf{C}$ and $\mathbf{b}$ (the eigenvalues of $\mathbf{b}$ are λ_i^2).

5.4.5. Euler-Almansi Strain Tensor

$$\mathbf{e} = \tfrac{1}{2}(\mathbf{I} - \mathbf{b}^{-1}) = \tfrac{1}{2}(\mathbf{I} - (\mathbf{F}\mathbf{F}^T)^{-1}).$$

Derivation: Starting from line elements in the spatial config, $d\mathbf{x}$ and using $d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$:

$$dS^2 = (d\mathbf{x})^T\mathbf{F}^{-T}\mathbf{F}^{-1}d\mathbf{x}, \quad ds^2 = d\mathbf{x}\cdot d\mathbf{x}.$$

So $ds^2 - dS^2 = d\mathbf{x}^T(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1})d\mathbf{x} = 2d\mathbf{x}^T\mathbf{e}\,d\mathbf{x}$, giving $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{b}^{-1})$.

Properties: - Symmetric, spatial-configuration tensor. - $\mathbf{e} = \mathbf{0}$ when $\mathbf{b} = \mathbf{I}$ (no deformation) ✓ - For rigid rotation, $\mathbf{e} = \mathbf{0}$ ✓

Relation to Green-Lagrange strain: The pull-back of $\mathbf{e}$ to the reference config is $\mathbf{E}$:

$$\mathbf{E} = \mathbf{F}^T\mathbf{e}\mathbf{F}.$$

5.4.6. Infinitesimal Strain

When deformations are small, $|\nabla_{\mathbf{X}}\mathbf{u}| \ll 1$, we drop the quadratic terms:

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}(\nabla_{\mathbf{X}}\mathbf{u} + (\nabla_{\mathbf{X}}\mathbf{u})^T) = \mathrm{sym}(\nabla_{\mathbf{X}}\mathbf{u}).$$

In this limit, $\mathbf{E} \approx \mathbf{e} \approx \boldsymbol{\varepsilon}$ (all three strain measures converge), and we no longer distinguish reference from spatial coordinates.

Index notation: $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$.

Note

Key distinction: $\mathbf{E}, \mathbf{e}$ are finite-strain measures (finite deformations). $\boldsymbol{\varepsilon}$ is the small-strain limit. For structures undergoing large deformations (rubber, metals in forming, large-strain plasticity), you must use $\mathbf{E}$ or $\mathbf{e}$ and the corresponding constitutive laws (L05, L06). For small deformations (most linear FEA), $\boldsymbol{\varepsilon}$ and additive decompositions are sufficient (L07–L09).

5.4.7. Piola and Cauchy Deformation Tensors (Reference)

Two related deformation tensors appear in some literature:

Piola deformation tensor: $\mathbf{B}_{\text{Piola}} = \mathbf{C}^{-1} = (\mathbf{F}^T\mathbf{F})^{-1}$. (Note: This is sometimes called the "inverse right Cauchy-Green" and should not be confused with the left Cauchy-Green $\mathbf{b}$.)

Cauchy deformation tensor: $\mathbf{c} = \mathbf{b}^{-1} = (\mathbf{F}\mathbf{F}^T)^{-1}$ (spatial). It is the inverse of the Finger tensor.

These are less commonly used in modern courses but appear in classical continuum mechanics texts.

5.4.8. Summary Table

Tensor	Formula	Configuration	Deformation-free value
Right Cauchy-Green	$\mathbf{C} = \mathbf{F}^T\mathbf{F}$	Reference	$\mathbf{I}$

Tensor	Formula	Configuration	Deformation-free value
Left Cauchy-Green / Finger	$\mathbf{b} = \mathbf{F}\mathbf{F}^T$	Current	$\mathbf{I}$
Green-Lagrange strain	$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$	Reference	$\mathbf{0}$
Almansi strain	$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{b}^{-1})$	Current	$\mathbf{0}$
Infinitesimal strain	$\boldsymbol{\varepsilon} = \mathrm{sym}(\nabla_{\mathbf{X}}\mathbf{u})$	Either (small-strain)	$\mathbf{0}$

5.5. Stretch and Rotation — Polar Decomposition

5.5.1. Theorem: Unique Polar Decompositions

Theorem. Every invertible deformation gradient $\mathbf{F}$ can be uniquely decomposed as:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R},$$

where: - $\mathbf{R}$ is an **orthogonal (rotation) tensor**, $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, $\det \mathbf{R} = +1$. - $\mathbf{U}$ is a **right stretch tensor** (material, symmetric positive-definite). - $\mathbf{V}$ is a **left stretch tensor** (spatial, symmetric positive-definite). - $\mathbf{R}$ is the same in both decompositions; $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$ (they are related by a rotation).

Proof sketch: The right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is symmetric and positive-definite. It therefore admits a unique symmetric positive-definite square root $\mathbf{U} = \sqrt{\mathbf{C}}$. Define $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. Then:

$$\mathbf{R}^T\mathbf{R} = (\mathbf{F}\mathbf{U}^{-1})^T(\mathbf{F}\mathbf{U}^{-1}) = \mathbf{U}^{-1}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-1} = \mathbf{U}^{-1}(\mathbf{U}^2)\mathbf{U}^{-1} = \mathbf{I}.$$

So $\mathbf{R}$ is orthogonal, $\det \mathbf{R} = \det \mathbf{F} / \det \mathbf{U} > 0$, thus $\det \mathbf{R} = +1$. Similarly, $\mathbf{U} = \mathbf{F}^T\mathbf{F}$ relates to $\mathbf{V}$ via $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$.

5.5.2. The Two Decompositions: Physical Interpretation

Right decomposition ($\mathbf{F} = \mathbf{R}\mathbf{U}$):

The material is **first stretched by U in the reference frame, then rotated by R into the current frame.**

Physically, imagine a material element: $\mathbf{U}$ captures the *shape change* (elongation, shearing) relative to the reference material axes. Then $\mathbf{R}$ captures the *rigid-body rotation* of those axes into the current spatial frame.

Left decomposition ($\mathbf{F} = \mathbf{V}\mathbf{R}$):

The material is **first rotated by R, then stretched by V in the current frame.**

Physically, the material first rotates (so the deforming axes align with the current frame), then stretches relative to the current directions.

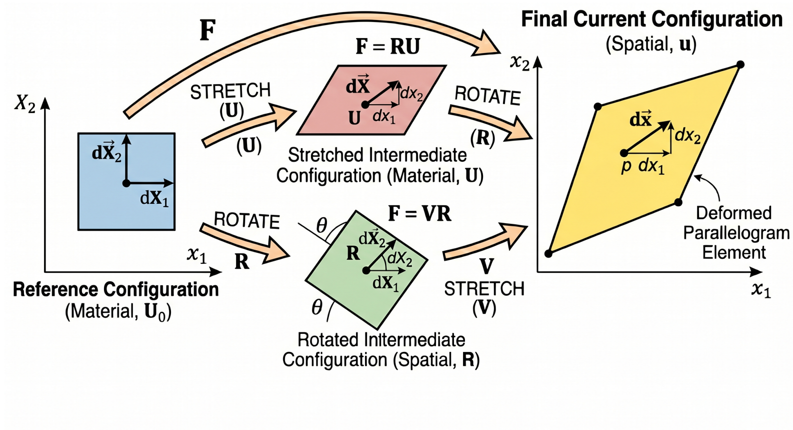


Figure 5.2.: The two polar decompositions of the deformation gradient $\mathbf{F}$. **Top path ($\mathbf{F} = \mathbf{RU}$):** the material is first stretched by $\mathbf{U}$ along principal directions $\mathbf{N}_i$ *in the reference configuration*, then rotated by $\mathbf{R}$ into its current orientation. **Bottom path ($\mathbf{F} = \mathbf{VR}$):** the material is first rotated by $\mathbf{R}$, then stretched by $\mathbf{V}$ along principal directions $\mathbf{n}_i = \mathbf{RN}_i$ *in the current configuration.* Both paths arrive at the same final state — the decomposition is a matter of *where* we choose to place the stretch relative to the rotation, not *what* the material actually does. The two stretches are similar tensors: $\mathbf{V} = \mathbf{RUR}^T$, and they share the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ — the principal stretches.

5.5.3. Relating the Two Stretches

$$\mathbf{V} = \mathbf{RUR}^T.$$

This shows that $\mathbf{U}$ and $\mathbf{V}$ are **similar tensors** — they have the same eigenvalues (principal stretches) but eigenvectors rotated by $\mathbf{R}$.

Relation to Cauchy-Green tensors:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2.$$

So we compute $\mathbf{U}$ by taking the symmetric square root of $\mathbf{C}$, and $\mathbf{V}$ by taking the symmetric square root of $\mathbf{b}$.

5.5.4. Principal Stretches

The **principal stretches** $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of both $\mathbf{U}$ and $\mathbf{V}$ (same values, but rotated directions):

$$\lambda_i^2 \text{ are the eigenvalues of } \mathbf{C} \text{ and } \mathbf{b}.$$

In the principal basis (aligned with the eigenvectors of $\mathbf{U}$):

$$\mathbf{U} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3), \quad \mathbf{C} = \mathrm{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2).$$

Physical interpretation: λ_i is the stretch ratio along the i-th principal direction. A material fiber initially aligned with the i-th principal axis is stretched by a factor λ_i. For a rigid rotation, $\lambda_i = 1$ for all i, so $\mathbf{U} = \mathbf{V} = \mathbf{I}$ and $\mathbf{F} = \mathbf{R}$.

5.5.5. Worked Example: Simple Shear Polar Decomposition

Consider again $\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (plane shear by γ).

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For small γ, the eigenvalues of $\mathbf{C}$ are approximately $\lambda_1^2 \approx 1 + \gamma^2, \lambda_2^2 \approx 1, \lambda_3^2 = 1$, giving $\lambda_1 \approx 1 + \gamma^2/2, \lambda_2 \approx 1, \lambda_3 = 1$. The deformation consists of a small stretch/compression in two directions and a rotation mixing them.

5.5.6. Spectral Decomposition

We can write $\mathbf{U}$ and $\mathbf{V}$ in spectral form:

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i, \quad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{n}_i \otimes \mathbf{n}_i,$$

where $\mathbf{N}_i$ are the principal directions in the reference config (eigenvectors of $\mathbf{C}$) and $\mathbf{n}_i$ are the principal directions in the current config (eigenvectors of $\mathbf{b}$). The rotation tensor maps them:

$$\mathbf{n}_i = \mathbf{R}\mathbf{N}_i.$$

5.6. Volume Changes and Isochoric Decomposition

5.6.1. The Jacobian as a Volume Ratio

As established earlier, $J = \det \mathbf{F}$ is the local volume ratio:

$$dv = J\, dV.$$

A material element at a point can change its volume (through hydrostatic pressure or incompressibility constraints) and its shape independently. For nearly-incompressible materials like rubber or plasticity at small volume change, it is useful to decompose the deformation into these parts.

5.6.2. Isochoric (Volume-Preserving) Decomposition

Define the **isochoric deformation gradient:**

$$\bar{\mathbf{F}} = J^{-1/3}\mathbf{F}.$$

Note that:

$$\det \bar{\mathbf{F}} = \det(J^{-1/3}\mathbf{F}) = (J^{-1/3})^3 \det \mathbf{F} = J^{-1}J = 1.$$

So $\bar{\mathbf{F}}$ represents a **volume-preserving deformation** — it captures pure shape change without volume change.

The original deformation can be reconstructed:

$$\mathbf{F} = J^{1/3}\bar{\mathbf{F}}.$$

Isochoric Cauchy-Green tensors:

$$\bar{\mathbf{C}} = \bar{\mathbf{F}}^T\bar{\mathbf{F}} = J^{-2/3}\mathbf{F}^T\mathbf{F} = J^{-2/3}\mathbf{C}, \quad \det \bar{\mathbf{C}} = 1.$$

Similarly for the left:

$$\bar{\mathbf{b}} = \bar{\mathbf{F}}\bar{\mathbf{F}}^T = J^{-2/3}\mathbf{b}, \quad \det \bar{\mathbf{b}} = 1.$$

5.6.3. Physical and Computational Utility

Many materials (elastomers, metals under pressure, plasticity) exhibit **decoupled volumetric-deviatoric response**: the stress can be split into a hydrostatic part (depending on volume change) and a deviatoric part (depending on shape change). With $\bar{\mathbf{C}}$ and $\bar{\mathbf{b}}$, we can write strain energies as:

$$W(\mathbf{C}) = W(\mathbf{C}, J) = W_{\text{iso}}(\bar{\mathbf{C}}) + W_{\text{vol}}(J),$$

where W_{iso} depends only on shape and W_{vol} only on volume. This separation is crucial for the hyperelastic formulations in L05 and plasticity in L07–L09.

5.7. Velocity Gradient and Strain Rate — Extensive Treatment

5.7.1. Definition of Velocity and Velocity Gradient

The **velocity field** is:

$$\mathbf{v}(\mathbf{x}, t) = \dot{\boldsymbol{\varphi}}(\mathbf{X}, t)$$

(the material time derivative of position).

The **velocity gradient** is the spatial gradient of velocity:

$$\mathbf{L} = \nabla_{\mathbf{x}} \mathbf{v} = \frac{\partial v_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j.$$

Fundamental relation: By the chain rule,

$$\dot{\mathbf{F}} = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{X}} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{L} \cdot \mathbf{F}.$$

Thus:

$$\mathbf{L} = \dot{\mathbf{F}} \mathbf{F}^{-1}.$$

This is a key identity: $\mathbf{L}$ is the time derivative of the deformation gradient in the spatial frame.

5.7.2. Symmetric and Skew Parts: Rate of Deformation and Spin

The velocity gradient decomposes uniquely into symmetric and skew-symmetric parts:

$$\mathbf{L} = \mathbf{D} + \mathbf{W},$$

where: - **Rate of deformation:** $\mathbf{D} = \operatorname{sym}\mathbf{L} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$. - **Spin (vorticity):** $\mathbf{W} = \operatorname{skew}\mathbf{L} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)$.

Index notation:

$$D_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad W_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right).$$

5.7.3. Physical Interpretation of $\mathbf{D}$ and $\mathbf{W}$

Rate of deformation D: Measures how line elements are stretching and shearing. If we follow a material line element $d\mathbf{x}(t)$, its rate of change is:

$$\frac{d}{dt}(d\mathbf{x}) = \mathbf{L}\,d\mathbf{x} = (\mathbf{D} + \mathbf{W})d\mathbf{x}.$$

The symmetric part $\mathbf{D}$ causes the length to change (through $\mathbf{D} : \mathbf{dx} \otimes d\mathbf{x}$) and angles to change (shear).

Spin W: A skew-symmetric tensor, it represents the local rotation rate. The axial vector of $\mathbf{W}$ is the **angular velocity $\boldsymbol{\omega}$**:

$$\mathbf{W}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$$

(i.e., $W_{ij}v_j = \varepsilon_{ijk}\omega_k v_j$). The spin describes how neighboring material elements rotate relative to one another.

5.7.4. Rate of Green-Lagrange Strain

$$\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{D}\mathbf{F}.$$

Derivation: Since $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$,

$$\dot{\mathbf{E}} = \tfrac{1}{2}(\dot{\mathbf{F}}^T\mathbf{F}+\mathbf{F}^T\dot{\mathbf{F}}) = \tfrac{1}{2}((\mathbf{L}\mathbf{F})^T\mathbf{F}+\mathbf{F}^T\mathbf{L}\mathbf{F}) = \tfrac{1}{2}(\mathbf{F}^T\mathbf{L}^T\mathbf{F}+\mathbf{F}^T\mathbf{L}\mathbf{F}) = \mathbf{F}^T\,\mathrm{sym}(\mathbf{L})\mathbf{F} = \mathbf{F}^T\mathbf{D}\mathbf{F}.$$

Note that only the symmetric part $\mathbf{D}$ contributes (the spin $\mathbf{W}$ does not).

5.7.5. Rate of Jacobian

$$\dot{J} = J\,\mathrm{tr}\,\mathbf{D} = J\,\mathrm{div}_{\mathbf{x}}\,\mathbf{v}.$$

Derivation: $J = \det\mathbf{F}$, so by the chain rule,

$$\dot{J} = \frac{\partial \det\mathbf{F}}{\partial \mathbf{F}} : \dot{\mathbf{F}} = (\mathrm{cof}\mathbf{F}) : \dot{\mathbf{F}} = J(\mathbf{F}^{-T} : \dot{\mathbf{F}}) = J\,\mathrm{tr}(\mathbf{F}^{-T}\dot{\mathbf{F}}) = J\,\mathrm{tr}(\mathbf{F}^{-1}\mathbf{L}\mathbf{F}) = J\,\mathrm{tr}\,\mathbf{L} = J\,\mathrm{tr}\,\mathbf{D}.$$

Since $\mathrm{tr}\,\mathbf{L} = \partial v_i/\partial x_i = \mathrm{div}\,\mathbf{v}$.

Physical meaning: $\mathrm{div}\,\mathbf{v} > 0$ means material is expanding (diverging flow field), $\mathrm{div}\,\mathbf{v} < 0$ means it is compressing. For an incompressible material, $\dot{J} = 0 \Rightarrow \mathrm{div}\,\mathbf{v} = 0$.

5.7.6. Cartesian Components

In Cartesian coordinates with unit basis:

$$L_{ij} = \frac{\partial v_i}{\partial x_j}, \quad D_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad W_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right).$$

These are commonly used in finite-strain numerical implementations.

5.7.7. Worked Example: Simple Shear Kinematics

Consider the shearing deformation $x_1 = X_1 + \gamma(t)X_2, x_2 = X_2, x_3 = X_3$ (shear strain increases with time).

Then $\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and the velocity is $\mathbf{v} = \dot{\mathbf{F}} X_2 \mathbf{e}_1 = \dot{\gamma} X_2 \mathbf{e}_1$ at $\mathbf{x} = (x_1, x_2, x_3)$.

In spatial coords, $v_1 = \dot{\gamma} x_2, v_2 = 0, v_3 = 0$.

$$\mathbf{L} = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & \dot{\gamma}/2 & 0 \\ \dot{\gamma}/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} 0 & \dot{\gamma}/2 & 0 \\ -\dot{\gamma}/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The strain rate $\mathbf{D}$ shows shear straining in the 1-2 plane; the spin $\mathbf{W}$ represents the rotation of material fibers (the skew part encodes rigid-body rotation of the material element).

5.8. Stress Measures — Four Measures and Transformations

5.8.1. Cauchy (True) Stress

The **Cauchy stress tensor** $\boldsymbol{\sigma}$ is defined via **Cauchy's theorem**: the traction (force per unit current area) on a surface with unit normal $\mathbf{n}$ in the current configuration is:

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}, \quad \text{or} \quad t_i = \sigma_{ij} n_j.$$

Properties: - Spatial (current-config) tensor, both indices lowercase. - Symmetric: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ (from angular momentum balance). - Measured in force per unit **current area**: if a patch of material in the current config has area da and normal $\mathbf{n}$, the total force is $\mathbf{t}\, da = \boldsymbol{\sigma}\mathbf{n}\, da$.

Sign convention (per course): Positive in tension, negative in compression. (Some texts use opposite signs.)

Physical interpretation: This is the "true" stress you would measure with a load cell in an experiment. As the material deforms, both the internal stress (force distribution) and the reference area change, so $\boldsymbol{\sigma}$ reflects the actual mechanical state in the deformed geometry.

5.8.2. First Piola-Kirchhoff Stress

The **1st Piola-Kirchhoff stress** (or **nominal stress**) $\mathbf{P}$ relates traction in the **reference configuration**. Define:

$$\mathbf{t}_0 = \mathbf{P}\mathbf{N}, \quad \text{or} \quad t_{0i} = P_{iJ}N_J.$$

where $\mathbf{N}$ is the unit normal and $\mathbf{t}_0$ is the traction on a reference-config surface element with area dA.

To derive the relation between $\mathbf{P}$ and $\boldsymbol{\sigma}$, use **Nanson's formula**:

$$\mathbf{n}\, da = J\mathbf{F}^{-T}\mathbf{N}\, dA.$$

Consider a reference surface element with traction $\mathbf{t}_0 dA$ (resulting in force $\mathbf{t}_0 dA$). After deformation, this force is still present but distributed over the current surface, so:

$$\mathbf{t}_0 dA = \boldsymbol{\sigma}\mathbf{n}\, da = \boldsymbol{\sigma}(J\mathbf{F}^{-T}\mathbf{N})\, dA.$$

Thus:

$$\mathbf{P}\mathbf{N} = J\boldsymbol{\sigma}\mathbf{F}^{-T}\mathbf{N} \quad \Rightarrow \quad \mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}.$$

Properties: - Two-point tensor: first index spatial (lowercase), second material (uppercase). - **Not symmetric in general**: $\mathbf{P} \neq \mathbf{P}^T$ (even though the material is in equilibrium, there is no angular momentum balance in the two-point space). - Measured in force per unit **reference area**.

Physical interpretation: $\mathbf{P}$ is useful for Lagrangian formulations because it refers to the known reference configuration. When you specify a displacement boundary condition in FEM, you are working in the reference config; stresses in that frame are naturally expressed via $\mathbf{P}$.

5.8.3. Second Piola-Kirchhoff Stress

The **2nd Piola-Kirchhoff stress S** is a **pure material-configuration stress** defined as:

$$\mathbf{S} = \mathbf{F}^{-1}\mathbf{P} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}.$$

Properties: - Material (reference) tensor, both indices uppercase. - **Symmetric**: $\mathbf{S} = \mathbf{S}^T$.

Why is it symmetric? Unlike $\mathbf{P}$, the second PK stress lives in the reference configuration alone, where the standard angular momentum argument applies. The pull-back operation $\mathbf{F}^{-1}(\cdot)\mathbf{F}^{-T}$ preserves symmetry.

Physical interpretation: Although $\mathbf{S}$ is defined mathematically via $\mathbf{P}$ and $\mathbf{F}$, it is the natural "conjugate" stress measure for the Green-Lagrange strain $\mathbf{E}$ in energy and power expressions (see section 7f below).

5.8.4. Kirchhoff Stress

The **Kirchhoff stress** is simply a rescaled Cauchy stress:

$$\boldsymbol{\tau} = J\boldsymbol{\sigma}.$$

Properties: - Spatial (current) tensor, but with a factor of J that "moves" it toward a material description. - Symmetric (since $\boldsymbol{\sigma}$ is).

Utility: In incompressible or nearly-incompressible material models, where volume is nearly conserved ($J \approx 1$), we have $\boldsymbol{\tau} \approx \boldsymbol{\sigma}$. The Kirchhoff stress simplifies many energy expressions and is standard in plasticity algorithms (L08).

5.8.5. Summary and Transformation Table

Measure	Symbol	Configuration	Symmetry	Conjugate strain-rate	Formula
Cauchy	$\boldsymbol{\sigma}$	Current	Yes	$\mathbf{D}$	(defined by Cauchy's theorem)
1st Piola-Kirchhoff	$\mathbf{P}$	Two-point	No	$\dot{\mathbf{F}}$	$J\boldsymbol{\sigma}\mathbf{F}^{-T}$
2nd Piola-Kirchhoff	$\mathbf{S}$	Reference	Yes	$\dot{\mathbf{E}}$	$J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$
Kirchhoff	$\boldsymbol{\tau}$	Current	Yes	$\mathbf{D}$	$J\boldsymbol{\sigma}$

Transformation formulas:

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}, \quad \boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{P}, \quad \mathbf{S} = \mathbf{F}^{-1}\mathbf{P} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}.$$

5.8.6. Worked Example: Uniaxial Loading

Consider the uniaxial tension deformation with $\lambda_1 = \lambda_2 = \lambda, \lambda_3 = 1/\lambda^2$ (constant volume), giving stretches $\mathbf{F} = \mathrm{diag}(\lambda, \lambda, 1/\lambda^2)$ and $J = 1$.

Suppose the material is linearly elastic with modulus E. The Cauchy stress is approximately $\boldsymbol{\sigma} \approx \begin{pmatrix} E(\lambda - 1/\lambda^4) & 0 & 0 \\ 0 & -0.5E(\lambda - 1/\lambda^4) & 0 \\ 0 & 0 & -0.5E(\lambda - 1/\lambda^4) \end{pmatrix}$ (simplified form for illustration).

The 1st PK becomes $\mathbf{P} = \boldsymbol{\sigma}\mathbf{F}^{-T} = \begin{pmatrix} E(\lambda - 1/\lambda^4)/\lambda & 0 & 0 \\ 0 & -0.5E(\lambda - 1/\lambda^4)/\lambda & 0 \\ 0 & 0 & -0.5E(\lambda - 1/\lambda^4)\lambda^2 \end{pmatrix}$ (showing the two-point nature: different scalings per component).

5.9. Stress-Strain Conjugates and Work Expressions

5.9.1. Power Conjugacy

In the energy balance, the rate of stress power (energy flux) per unit material volume is:

$$P_{\text{int}} = \int_{\Omega_t} \boldsymbol{\sigma} : \mathbf{D}\, dv = \int_{\Omega_0} \mathbf{P} : \dot{\mathbf{F}}\, dV = \int_{\Omega_0} \mathbf{S} : \dot{\mathbf{E}}\, dV = \int_{\Omega_t} \boldsymbol{\tau} : \mathbf{D}\, dv.$$

All four expressions are **equivalent** and equal the rate of internal energy (per unit time). The pairs $(\boldsymbol{\sigma}, \mathbf{D})$**,** $(\mathbf{P}, \dot{\mathbf{F}})$**,** $(\mathbf{S}, \dot{\mathbf{E}})$**, and** $(\boldsymbol{\tau}, \mathbf{D})$ **are called work-conjugate pairs**.

Derivation (sketch): Start with $P_{\text{int}} = \int \boldsymbol{\sigma} : \mathbf{D}\, dv$. Using $\mathbf{D} = \mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}$ (pull-back) and $dv = JdV$:

$$\int \boldsymbol{\sigma} : \mathbf{D}\, dv = \int \boldsymbol{\sigma} : (\mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}) J\, dV.$$

Rearranging using cyclic permutation of the double contraction:

$$\boldsymbol{\sigma} : (\mathbf{F}^{-T}\dot{\mathbf{E}}\mathbf{F}^{-1}) = (J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}) : \dot{\mathbf{E}} = \mathbf{S} : \dot{\mathbf{E}}.$$

Thus, $\int \boldsymbol{\sigma} : \mathbf{D}\, dv = \int \mathbf{S} : \dot{\mathbf{E}}\, dV$ ✓

5.9.2. Implications for Constitutive Laws

In a hyperelastic material, the Helmholtz free energy Ψ per unit reference volume is a function of the deformation gradient (or strain). The stress is derived from this energy:

$$\mathbf{S} = \rho_0 \frac{\partial \Psi}{\partial \mathbf{E}}, \quad \boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^T = J^{-1}\rho_0 \mathbf{F}\frac{\partial \Psi}{\partial \mathbf{E}}\mathbf{F}^T.$$

The choice of **which pair to use** depends on the formulation: - **Lagrangian FEM (material-frame):** Use $(\mathbf{S}, \dot{\mathbf{E}})$ or $(\mathbf{P}, \dot{\mathbf{F}})$. - **Eulerian FEM (spatial-frame):** Use $(\boldsymbol{\sigma}, \mathbf{D})$ or $(\boldsymbol{\tau}, \mathbf{D})$. - **Plasticity (usually Lagrangian):** Use $(\mathbf{S}, \dot{\mathbf{E}})$ or $(\boldsymbol{\sigma}, \mathbf{D})$ depending on the strain measure.

The equivalence of the power expressions ensures thermodynamic consistency across all formulations.

5.10. Objectivity of Stress Rates

Material frame invariance: Under a superimposed rigid-body motion $\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t)$ (where $\mathbf{Q}$ is orthogonal and $\mathbf{c}$ is a translation):

- **Cauchy stress $\boldsymbol{\sigma}$ is objective: $\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T$.**
- **The material time derivative $\dot{\boldsymbol{\sigma}}$ is NOT objective** — it changes under rigid rotation.
- **2nd Piola-Kirchhoff S is objective** (lives in the material frame, which is unrotated).

- **The rate $\dot{\mathbf{S}}$ is also objective** (both are material-frame quantities).

This asymmetry is crucial: when writing rate-form constitutive equations like $\dot{\boldsymbol{\sigma}} = \mathbb{C} : \mathbf{D}$, the equation is not objective unless additional rotation-correction terms are added. Common objective stress rates are:

- **Jaumann rate:** $\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W}$ (corrects for local spin $\mathbf{W}$).
- **Green-Naghdi rate, Truesdell rate** (alternative definitions; see L02 for detailed proofs).

Recommendation: For any rate-form hypoelastic model (e.g., plasticity), use an objective rate of stress. The second Piola-Kirchhoff $\mathbf{S}$ is automatically objective; if using Cauchy stress, apply a rate correction. See L02 for rigorous treatment and L08 for algorithmic implementation in plasticity.

5.11. Conservation Laws

Mass: $\dot{\rho} + \rho\nabla \cdot \mathbf{v} = 0$ (spatial), or $\rho J = \rho_0$ (material).

Linear momentum:

$$\nabla \cdot \boldsymbol{\sigma} + \rho\mathbf{b} = \rho\ddot{\mathbf{x}}$$

Angular momentum: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ (symmetry of Cauchy stress).

Energy: First law of thermodynamics (developed in L04).

5.12. Preliminaries: Material Derivative

The material derivative of the Jacobian determinant of the deformation gradient $\mathbf{F}$:

$$\frac{DJ}{Dt} = J\,\mathrm{div}(\mathbf{v}) = J\frac{\partial v_i}{\partial x_i}$$

The material derivative of an integral over a moving domain is:

$$\frac{D}{Dt}\int_{\Omega} f(\mathbf{x},t)d\Omega = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{\Omega_{t+\Delta t}} f(\mathbf{x},t+\Delta t)d\Omega - \int_{\Omega_t} f(\mathbf{x},t)d\Omega\right)$$

where Ω_t is the spatial domain at time t and $f(\mathbf{x},t)$ is a function defined on that domain.

We can express the right-hand side in the reference configuration using the Jacobian of the deformation gradient:

$$\frac{D}{Dt}\int_{\Omega} f d\Omega = \int_{\Omega_0}\frac{\partial}{\partial t}[f(\mathbf{X},t)J(\mathbf{X},t)]d\Omega_0$$

After rearranging:

$$\frac{D}{Dt}\int_{\Omega} f d\Omega = \int_{\Omega_0}\left(\frac{\partial f}{\partial t}J + f\frac{\partial J}{\partial t}\right)d\Omega_0 = \int_{\Omega_0}\left(\frac{\partial f}{\partial t}J + fJ\frac{\partial v_i}{\partial x_i}\right)d\Omega_0$$

Transforming back to the current configuration yields **Reynolds' Transport Theorem**:

$$\frac{D}{Dt}\int_{\Omega} f d\Omega = \int_{\Omega}\left(\frac{Df(\mathbf{x},t)}{Dt} + f\frac{\partial v_i}{\partial x_i}\right)d\Omega$$

which can also be written as:

$$\frac{D}{Dt}\int_{\Omega} f d\Omega = \int_{\Omega}\left(\frac{\partial f}{\partial t} + \mathrm{div}(\mathbf{v}f)\right)d\Omega$$

or using Gauss's theorem:

$$\frac{D}{Dt}\int_{\Omega} f d\Omega = \int_{\Omega}\frac{\partial f}{\partial t}d\Omega + \int_{\Gamma} f\mathbf{v}\cdot\mathbf{n}d\Gamma$$

where Γ is the boundary of the domain Ω and $\mathbf{n}$ is the outward normal vector.

5.13. Conservation of Linear Momentum (Detailed)

The total force on a system is given by:

$$\mathbf{f}(t) = \int_{\Omega} \rho \mathbf{b}(\mathbf{x}, t) d\Omega + \int_{\Gamma} \mathbf{t}(\mathbf{x}, t) d\Gamma$$

where **b** is the body force per unit mass and **t** is the traction vector. The linear momentum is:

$$\mathbf{p}(t) = \int_{\Omega} \rho \mathbf{v}(\mathbf{x}, t) d\Omega$$

Newton's second law states:

$$\frac{D\mathbf{p}}{Dt} = \mathbf{f}(t) \Rightarrow \frac{D}{Dt} \int_{\Omega} \rho \mathbf{v}(\mathbf{x}, t) d\Omega = \int_{\Omega} \rho \mathbf{b}(\mathbf{x}, t) d\Omega + \int_{\Gamma} \mathbf{t}(\mathbf{x}, t) d\Gamma$$

Using Reynolds' theorem and Gauss's theorem with $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$:

$$\int_{\Gamma} \mathbf{t} d\Gamma = \int_{\Gamma} \boldsymbol{\sigma}\mathbf{n} d\Gamma = \int_{\Omega} \operatorname{div}(\boldsymbol{\sigma}) d\Omega$$

After manipulation we arrive at the local form:

$$\rho \frac{D\mathbf{v}}{Dt} = \operatorname{div}(\boldsymbol{\sigma}) + \rho \mathbf{b}$$

This is the equation of motion (momentum balance) in spatial form.

5.14. Conservation of Energy (First Law of Thermodynamics)

For thermomechanical processes, the rate of change of total energy is:

$$P^{tot} = P^{int} + P^{kin}$$

with:

$$P^{int} = \frac{D}{Dt}\int_\Omega \rho\omega^{int} d\Omega, \quad P^{kin} = \frac{D}{Dt}\int_\Omega \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} d\Omega$$

where ω^{int} is the specific internal energy and ω^{kin} is the specific kinetic energy.

The rate of external work is:

$$P^{ext} = \int_\Gamma \mathbf{v}\cdot\mathbf{t} d\Gamma + \int_\Omega \mathbf{v}\cdot\rho\mathbf{b} d\Omega$$

And heat sources contribute:

$$P^{heat} = \int_\Omega \rho s d\Omega - \int_\Gamma \mathbf{n}\cdot\mathbf{q} d\Gamma$$

where $\mathbf{q}$ is the heat flux vector and s is the volumetric heat source per unit mass.

The **Total Energy Balance** is:

$$P^{tot} = P^{ext} + P^{heat}$$

After applying Reynolds' theorem, Gauss' theorem, and the momentum balance equation, we obtain the **energy equation in spatial form**:

$$\rho\frac{D\omega^{int}}{Dt} = \mathbf{d} : \boldsymbol{\sigma} - \mathrm{div}(\mathbf{q}) + \rho s$$

where $\mathbf{d}$ is the rate of deformation tensor and ω^{int} is the internal energy per unit mass.

In **Lagrangian form** (material configuration), using the first Piola-Kirchhoff stress $\mathbf{P}$ and material (reference) gradient ∇_0:

$$\rho_0\dot{\omega}^{int} = \dot{\mathbf{F}}^T : \mathbf{P} - \nabla_0\cdot\mathbf{q} + \rho_0 s$$

where ρ_0 is the reference density.

5.15. Strong Form and Weak Form: Principle of Virtual Work

We start with the **static equilibrium equation (strong form)** in domain Ω:

$$\mathrm{div}(\boldsymbol{\sigma}) = \mathbf{0} \quad \text{in } \Omega$$

Subject to boundary conditions: - Essential (Dirichlet) BCs: $\mathbf{u} = \bar{\mathbf{u}}$ on Γ_u - Natural (Neumann) BCs: $\boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}$ on Γ_t

For nonlinear materials, stress is a nonlinear function of strain:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\varepsilon(\nabla \mathbf{u}))$$

To derive the **weak form**, multiply the equilibrium equation by a virtual displacement field $\delta\mathbf{u}$ (satisfying $\delta\mathbf{u} = \mathbf{0}$ on Γ_u) and integrate:

$$\int_\Omega \mathrm{div}(\boldsymbol{\sigma}) \cdot \delta\mathbf{u}\, dv = 0$$

Using the divergence theorem and applying boundary conditions, we obtain the **Principle of Virtual Work**:

$$\int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, dv = \int_{\Gamma_t} \bar{\mathbf{t}} \cdot \delta\mathbf{u}\, da$$

where $\delta\boldsymbol{\varepsilon} = \mathrm{sym}(\nabla(\delta\mathbf{u}))$ is the virtual strain.

This equation holds for all admissible virtual displacements: **internal virtual work = external virtual work**.

5.16. Finite Element Discretization

We discretize the domain into elements Ω_e. Within each element, we approximate the displacement using shape functions $\mathbf{N}(\mathbf{x})$ and nodal displacements $\mathbf{d}_e$:

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{N}(\mathbf{x})\mathbf{d}_e \quad \Rightarrow \quad \boldsymbol{\varepsilon}(\mathbf{x}) \approx \mathbf{B}(\mathbf{x})\mathbf{d}_e$$

Similarly for virtual fields:

$$\delta\mathbf{u}(\mathbf{x}) = \mathbf{N}(\mathbf{x})\delta\mathbf{d}_e \quad \Rightarrow \quad \delta\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\delta\mathbf{d}_e$$

where $\mathbf{B}$ is the strain-displacement matrix (containing spatial derivatives of $\mathbf{N}$).

Substituting into the weak form and assembling over all elements:

$$\sum_e \int_{\Omega_e} \mathbf{B}^T \boldsymbol{\sigma}(\mathbf{B}\mathbf{d}_e)\, dv = \sum_e \int_{\Gamma_{t,e}} \mathbf{N}^T \bar{\mathbf{t}}\, da$$

This defines: - **Internal Force Vector:** $\mathbf{f}_{\text{int}}(\mathbf{d}) = \text{Assembly}\left(\int_{\Omega_e} \mathbf{B}^T \boldsymbol{\sigma}(\boldsymbol{\varepsilon}(\mathbf{d}))\, dv\right)$
- **External Force Vector:** $\mathbf{f}_{\text{ext}} = \text{Assembly}\left(\int_{\Gamma_{t,e}} \mathbf{N}^T \bar{\mathbf{t}}\, da\right)$

The discrete equilibrium equation is:

$$\mathbf{f}_{\text{int}}(\mathbf{d}) = \mathbf{f}_{\text{ext}}$$

or equivalently, find the root of the **residual vector**:

$$\mathbf{R}(\mathbf{d}) = \mathbf{f}_{\text{int}}(\mathbf{d}) - \mathbf{f}_{\text{ext}} = \mathbf{0}$$

5.17. Nonlinearity in FEM

Material Nonlinearity: If the material law $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ is linear, the stiffness matrix is constant and we have a linear system $\mathbf{Kd} = \mathbf{f}_{\text{ext}}$.

If the material law is nonlinear $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon})$, then $\mathbf{f}_{\text{int}}(\mathbf{d})$ is a nonlinear function of $\mathbf{d}$, leading to a system of nonlinear algebraic equations.

Geometric Nonlinearity: For small deformations, we use the infinitesimal strain:

$$\boldsymbol{\varepsilon}_{\text{lin}} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

where $\epsilon_{\text{lin}} \approx \mathbf{B}_{\text{lin}}\mathbf{d}$ is linear in $\mathbf{d}$.

For large deformations, we use the Green-Lagrange strain:

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \underbrace{\frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)}_{\text{linear}} + \underbrace{\frac{1}{2}(\nabla\mathbf{u})^T(\nabla\mathbf{u})}_{\text{quadratic term}}$$

The quadratic term is negligible for small deformations but essential for large deformations.

After FE discretization with $\nabla\mathbf{u} \approx \mathbf{G}\mathbf{d}_e$:

$$\mathbf{E} \approx \underbrace{\frac{1}{2}(\mathbf{G}\mathbf{d}_e + (\mathbf{G}\mathbf{d}_e)^T)}_{\text{linear in } \mathbf{d}_e} + \underbrace{\frac{1}{2}(\mathbf{G}\mathbf{d}_e)^T(\mathbf{G}\mathbf{d}_e)}_{\text{quadratic in } \mathbf{d}_e}$$

The strain becomes a **nonlinear (quadratic) function** of nodal displacements, often written as:

$$\boldsymbol{\varepsilon}_{\text{NL}} \approx (\mathbf{B}_0(\mathbf{x}) + \mathbf{B}_L(\mathbf{x}, \mathbf{d}_e))\mathbf{d}_e$$

Consequences: - Even with a linear material law, geometric nonlinearity makes $\mathbf{f}_{\text{int}}(\mathbf{d})$ nonlinear because strain depends nonlinearly on $\mathbf{d}$. - The **tangent stiffness matrix** $\mathbf{K}_T = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{d}}$ includes both material and **geometric stiffness** contributions.

5.18. Newton-Raphson Iteration

To solve the nonlinear system $\mathbf{R}(\mathbf{d}) = \mathbf{0}$, we use Newton-Raphson iteration.

Starting from an estimate $\mathbf{d}_i$, we seek a correction $\Delta\mathbf{d}$ such that $\mathbf{R}(\mathbf{d}_i + \Delta\mathbf{d}) \approx \mathbf{0}$.

Using a Taylor expansion:

$$\mathbf{R}(\mathbf{d}_i + \Delta\mathbf{d}) \approx \mathbf{R}(\mathbf{d}_i) + \left[\frac{\partial \mathbf{R}}{\partial \mathbf{d}}\right]_{\mathbf{d}_i} \Delta\mathbf{d}$$

Setting the approximation to zero gives the iterative update:

$$\left[\frac{\partial \mathbf{R}}{\partial \mathbf{d}}\right]_{\mathbf{d}_i} \Delta\mathbf{d} = -\mathbf{R}(\mathbf{d}_i)$$

The Jacobian matrix is the **Tangent Stiffness Matrix**:

$$\mathbf{K}_T(\mathbf{d}_i) = \left[\frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{d}}\right]_{\mathbf{d}_i}$$

Differentiating the internal force expression:

$$\mathbf{K}_T \approx \text{Assembly} \int_{\Omega_e} \mathbf{B}^T \left(\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}\right) \mathbf{B}\, dv$$

where $\mathbf{D}_T = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}$ is the **material tangent modulus**, evaluated at the current strain state.

Newton-Raphson algorithm:

1. Calculate residual: $\mathbf{R}_i = \mathbf{f}_{\text{int}}(\mathbf{d}_i) - \mathbf{f}_{\text{ext}}$
2. Calculate tangent stiffness: $\mathbf{K}_T(\mathbf{d}_i)$
3. Solve linear system: $\mathbf{K}_T(\mathbf{d}_i)\Delta\mathbf{d} = -\mathbf{R}_i$
4. Update solution: $\mathbf{d}_{i+1} = \mathbf{d}_i + \Delta\mathbf{d}$
5. Repeat until convergence (e.g., $\|\Delta\mathbf{d}\| < \varepsilon$)

6. L04 — Thermodynamic Framework

Laws of Thermodynamics, Dissipation, and the FEM Stress-Update Context

Sign convention — heat flux. The heat flux vector $\mathbf{q}$ points in the direction of **positive heat flow** (outward from hot to cold). In the energy-balance equation, the term $-\nabla \cdot \mathbf{q}$ represents heat **supplied to** the body (so a net positive divergence of $\mathbf{q}$ means heat is leaving).

6.1. First Law of Thermodynamics

The rate of change of total energy equals the power input plus heat supply:

$$\dot{E} = \mathcal{P}_{\text{ext}} + \mathcal{Q},$$

In local (strong) form with internal energy density e:

$$\rho \dot{e} = \boldsymbol{\sigma} : \mathbf{d} + \rho r - \nabla \cdot \mathbf{q}$$

where r is the heat source density and $\mathbf{q}$ the heat flux.

6.2. Second Law — Entropy Inequality

The Clausius-Duhem inequality:

$$\rho\dot{\eta} \geq \frac{\rho r}{\theta} - \nabla \cdot \left(\frac{\mathbf{q}}{\theta}\right)$$

where η is the specific entropy and $\theta > 0$ the absolute temperature.

Introducing the **Helmholtz free energy** $\Psi = e - \theta\eta$ and substituting:

$$\mathcal{D} = \boldsymbol{\sigma} : \mathbf{d} - \rho\dot{\Psi} - \rho\eta\dot{\theta} - \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta \geq 0$$

This is the **dissipation inequality** — the cornerstone for thermodynamically consistent constitutive models.

6.3. Isothermal Conditions and Free Energy Potentials

For isothermal processes ($\dot{\theta} = 0$, $\nabla\theta = \mathbf{0}$):

$$\mathcal{D} = \boldsymbol{\sigma} : \mathbf{d} - \rho\dot{\Psi} \geq 0$$

The free energy Ψ depends on:

- **Elastic strain** (recoverable energy storage)
- **Internal variables $\boldsymbol{\alpha}$** (irreversible processes: plastic strain, damage, etc.)

6.4. Constitutive Restrictions from Thermodynamics

For a **hyperelastic** material in material description, $\Psi = \hat{\Psi}(\mathbf{C})$:

$$\mathbf{S} = 2\rho_0 \frac{\partial \hat{\Psi}}{\partial \mathbf{C}}$$

This guarantees $\mathcal{D} = 0$ (purely elastic, no dissipation).

For **inelastic** materials, the free energy depends on elastic strains **and** internal variables:

$$\Psi = \hat{\Psi}(\mathbf{C}^e, \boldsymbol{\alpha})$$

Dissipation $\mathcal{D} \geq 0$ constrains the evolution equations for $\boldsymbol{\alpha}$.

> **Notation — `$\mathbf{C}^e$`.** In this chapter, `$\mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e$` is the **elastic part of the right Cauchy-Green tensor** (a second-order kinematic quantity, from the multiplicative split $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$). This is distinct from the **fourth-order elasticity tensor** `$\mathbb{C}^e$` (blackboard bold) used in L07–L09 plasticity chapters. Typography distinguishes them: $\mathbf{C}^e$ is bold upright (2nd-order), $\mathbb{C}^e$ is blackboard (4th-order).

6.5. The Stress-Update Problem in FEM

In a nonlinear FEM analysis, at each time/load step the constitutive law is called at each Gauss point to:

1. **Input:** strain increment $\Delta\boldsymbol{\varepsilon}$ (from displacement increment), previous stress $\boldsymbol{\sigma}_n$, previous internal variables $\boldsymbol{\alpha}_n$.
2. **Output:** updated stress $\boldsymbol{\sigma}_{n+1}$, updated internal variables $\boldsymbol{\alpha}_{n+1}$, consistent tangent modulus $\mathbb{C}_{n+1}$.

The **consistent (algorithmic) tangent** $\mathbb{C} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}$ is required for the global Newton-Raphson loop to converge quadratically.

6.6. Generalized Standard Materials

A material is a **Generalized Standard Material** (GSM, Halphen & Son 1975) if:

- A free energy $\Psi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha})$ governs reversible response.
- A convex dissipation potential $\phi(\dot{\boldsymbol{\alpha}})$ governs irreversible evolution:

$$\boldsymbol{A} = -\frac{\partial \Psi}{\partial \boldsymbol{\alpha}}, \qquad \dot{\boldsymbol{\alpha}} \in \partial_{\boldsymbol{A}} \phi^*(\boldsymbol{A})$$

Most standard models (J2 plasticity, linear isotropic damage) fit this framework and inherit thermodynamic consistency automatically.

6.7. Material Frame-Indifference (Objectivity)

The principle of **material frame-indifference** states that the constitutive law of a material must be independent of the observer.

The material's response should not depend on whether the observer is stationary, translating at constant velocity, or rotating relative to the material.

Mathematically, if we apply a rigid body motion (superimposed translation and rotation) to the entire system, the form of the constitutive equation must remain the same when expressed in terms of the transformed variables in the new frame.

For a quantity like the Cauchy stress tensor $\boldsymbol{\sigma}$ and deformation gradient $\mathbf{F}$, transformation under a change of frame with time-dependent rotation tensor $\mathbf{Q}(t)$ gives:

$$\boldsymbol{\sigma}^* = \mathbf{Q}(t)\boldsymbol{\sigma}\mathbf{Q}(t)^T, \quad \mathbf{F}^* = \mathbf{Q}(t)\mathbf{F}$$

Frame-indifference requires: if $\boldsymbol{\sigma} = \mathbb{F}(\mathbf{F}, \ldots)$, then $\boldsymbol{\sigma}^* = \mathbb{F}(\mathbf{F}^*, \ldots)$, leading to:

$$\mathbf{Q}(t)\mathbb{F}(\mathbf{F}, \ldots)\mathbf{Q}(t)^T = \mathbb{F}(\mathbf{Q}(t)\mathbf{F}, \ldots)$$

This must hold for all orthogonal tensors $\mathbf{Q}(t)$.

Consequence: The free energy cannot depend on $\mathbf{F}$ directly, but only through objective combinations like: - Right Cauchy-Green tensor: $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ - Green-Lagrange strain: $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ - Left stretch tensor: $\mathbf{U}$ from polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$

So we write: $\Psi = \tilde{\Psi}(\mathbf{C}, \theta)$ or $\Psi = \hat{\Psi}(\mathbf{E}, \theta)$ instead of $\Psi = \Psi(\mathbf{F}, \theta)$.

6.8. Coleman-Noll Procedure

The **Coleman-Noll procedure** is a systematic method to derive restrictions on constitutive equations by combining:

1. **Material Frame-Indifference** (objectivity)
2. **Second Law of Thermodynamics** (Clausius-Duhem inequality)

The Key Idea:

The procedure exploits the fact that superimposed rigid motions (especially rotations) can be chosen *arbitrarily* at any instant. By: - Assuming a general functional form for constitutive relations - Applying the mathematical statement of frame-indifference - Choosing specific, convenient rotations to simplify - Invoking arbitrariness of thermodynamic rates

We can deduce necessary conditions on the form of material functions.

For Hyperelastic Materials:

Assume the thermodynamic state is determined by $\mathbf{F}$ and θ. We postulate:

$$\Psi = \hat{\Psi}(\mathbf{F}, \theta), \quad \boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{F}, \theta), \quad \eta = \hat{\eta}(\mathbf{F}, \theta)$$

Using the dissipation inequality (neglecting heat flux):

$$\mathbf{P} : \dot{\mathbf{F}} - \rho_0(\dot{\Psi} + \eta\dot{\theta}) \geq 0$$

where $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$ is the first Piola-Kirchhoff stress.

From objectivity: $\Psi = \tilde{\Psi}(\mathbf{C}, \theta)$ with $\dot{\mathbf{C}} = \dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}}$.

The rate of free energy is:

$$\dot{\Psi} = \frac{\partial\tilde{\Psi}}{\partial\mathbf{C}} : \dot{\mathbf{C}} + \frac{\partial\tilde{\Psi}}{\partial\theta}\dot{\theta}$$

Substituting into the dissipation inequality:

$$\mathbf{P} : \dot{\mathbf{F}} - \rho_0\left(\frac{\partial\tilde{\Psi}}{\partial\mathbf{C}} : (\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}}) + \frac{\partial\tilde{\Psi}}{\partial\theta}\dot{\theta} + \eta\dot{\theta}\right) \geq 0$$

Using tensor identities:

$$\frac{\partial\tilde{\Psi}}{\partial\mathbf{C}} : (\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}}) = 2\left(\mathbf{F}\frac{\partial\tilde{\Psi}}{\partial\mathbf{C}}\right) : \dot{\mathbf{F}}$$

The inequality becomes:

$$\left(\mathbf{P} - 2\rho_0\mathbf{F}\frac{\partial\tilde{\Psi}}{\partial\mathbf{C}}\right) : \dot{\mathbf{F}} - \rho_0\left(\frac{\partial\tilde{\Psi}}{\partial\theta} + \eta\right)\dot{\theta} \geq 0$$

6.9. Coleman-Noll Argument: Exploiting Arbitrariness

This inequality must hold for *any* kinematically and thermodynamically admissible process—meaning *arbitrary* $\dot{\mathbf{F}}$ and $\dot{\theta}$.

For the temperature rate $\dot{\theta}$:

If the coefficient $\rho_0(\frac{\partial\tilde{\Psi}}{\partial\theta} + \eta)$ were non-zero, we could choose $\dot{\mathbf{F}} = \mathbf{0}$ and select $\dot{\theta}$ with opposite sign to violate the inequality.

Therefore, the coefficient must be zero:

$$\eta = -\frac{\partial\tilde{\Psi}}{\partial\theta}$$

This is the standard **thermodynamic relation for entropy**.

For the deformation gradient rate $\dot{\mathbf{F}}$:

With the temperature term now zero, we need:

$$\left(\mathbf{P} - 2\rho_0\mathbf{F}\frac{\partial\tilde{\Psi}}{\partial\mathbf{C}}\right) : \dot{\mathbf{F}} \geq 0$$

For a non-dissipative elastic material, this coefficient must also be zero:

$$\mathbf{P} = 2\rho_0\mathbf{F}\frac{\partial\tilde{\Psi}}{\partial\mathbf{C}}$$

Converting to Cauchy Stress:

Using the relation $\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^T = (\rho/\rho_0)\mathbf{P}\mathbf{F}^T$:

$$\boldsymbol{\sigma} = 2\rho\mathbf{F}\frac{\partial\tilde{\Psi}(\mathbf{C},\theta)}{\partial\mathbf{C}}\mathbf{F}^T$$

Alternatively, using Green-Lagrange strain $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$:

$$\boldsymbol{\sigma} = \rho \mathbf{F} \frac{\partial \hat{\Psi}(\mathbf{E}, \theta)}{\partial \mathbf{E}} \mathbf{F}^T$$

Stress is derived from a potential (the free energy) by differentiation with respect to strain.

6.10. Significance of Coleman-Noll Results

The Coleman-Noll procedure provides:

1. **Rigorous Foundation:** Constitutive models are guaranteed to be consistent with fundamental physical laws (objectivity and thermodynamics).
2. **Restricted Possibilities:** The set of admissible material model forms is greatly reduced, guiding rational development.
3. **Identification of Variables:** Shows that material response depends on objective strain measures (like $\mathbf{C}$ or $\mathbf{E}$), not the full deformation gradient.
4. **Hyperelasticity Condition:** A material is hyperelastic if and only if stress can be derived from a free energy potential—no independent dissipation beyond the potential itself.
5. **Entropy Relations:** Entropy is not an independent variable but determined by free energy derivatives.
6. **Inelastic Dissipation:** When inelastic processes occur (plasticity, damage), they appear as *additional* dissipation terms beyond zero, constrained to be non-negative by the dissipation inequality. Evolution equations for internal variables must be chosen to maintain $\mathcal{D} \geq 0$.

6.11. Mathematical Identity: Chain Rule for Free Energy

Goal: Show that the free energy rate depends on strain rate via a specific relationship with the deformation gradient.

To prove:

$$\frac{\partial\tilde{\Psi}}{\partial\mathbf{C}} : \dot{\mathbf{C}} = \left(2\mathbf{F}\frac{\partial\tilde{\Psi}}{\partial\mathbf{C}}\right) : \dot{\mathbf{F}}$$

Key Properties: - $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ (symmetric) - $\dot{\mathbf{C}} = \dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}}$ (symmetric) - $\mathbf{S}_\psi = \frac{\partial\tilde{\Psi}}{\partial\mathbf{C}}$ is symmetric - Tensor trace cyclic property: $\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}(\mathbf{CAB})$

Proof: Start with LHS using $\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B})$:

$$\mathrm{LHS} = \mathbf{S}_\psi : (\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}})$$

For the first term, using trace cyclic property and symmetry:

$$\mathbf{S}_\psi : (\dot{\mathbf{F}}^T\mathbf{F}) = \mathrm{tr}(\mathbf{S}_\psi\dot{\mathbf{F}}^T\mathbf{F}) = \mathrm{tr}(\mathbf{F}\mathbf{S}_\psi\dot{\mathbf{F}}^T) = (\mathbf{F}\mathbf{S}_\psi) : \dot{\mathbf{F}}$$

For the second term:

$$\mathbf{S}_\psi : (\mathbf{F}^T\dot{\mathbf{F}}) = \mathrm{tr}(\mathbf{F}\mathbf{S}_\psi\dot{\mathbf{F}}^T) = (\mathbf{F}\mathbf{S}_\psi) : \dot{\mathbf{F}}$$

Adding both terms:

$$\mathrm{LHS} = 2(\mathbf{F}\mathbf{S}_\psi) : \dot{\mathbf{F}} = \left(2\mathbf{F}\frac{\partial\tilde{\Psi}}{\partial\mathbf{C}}\right) : \dot{\mathbf{F}} \quad \checkmark$$

Part II.

Part II — Constitutive Models

7. L05 — Hyperelasticity

Strain Energy Functions, Isotropic Models, and Numerical Implementation

7.1. What is Hyperelasticity?

A material is **hyperelastic** (Green elastic) if there exists a **strain energy function** (SEF) $W = \hat{W}(\mathbf{F})$ such that:

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \qquad \mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}}.$$

This automatically satisfies the Clausius-Duhem inequality (no dissipation). Frame indifference restricts W to depend on $\mathbf{F}$ only through $\mathbf{C} = \mathbf{F}^T\mathbf{F}$.

> **Notation — reference vs current configuration.** Following the course convention (see L03), uppercase symbols denote quantities defined on the **reference** (Lagrangian) configuration and lowercase symbols denote quantities defined on the **current** (Eulerian) configuration. In particular, the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is uppercase because it is a reference-frame quantity (pull-back of the spatial metric), while the **left Cauchy-Green / Finger tensor** $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ is lowercase because it is a current-frame quantity (push-

forward of the reference metric). The two share the same principal invariants but live on different configurations.

Fourth-order elasticity tensors. The material (Lagrangian) tangent is $\mathbb{D} = 2\,\partial\mathbf{S}/\partial\mathbf{C}$; its push-forward to the current configuration is (lowercase blackboard). Both are fourth-order. In L07–L09 plasticity chapters the 4th-order elasticity tensor is denoted $\mathbb{C}^e$ (material); that is the small-strain specialisation of $\mathbb{D}$.

7.2. Isotropy Restriction

For an **isotropic** hyperelastic material, W must depend only on the principal invariants of $\mathbf{C}$ (or equivalently $\mathbf{b}$):

$$W = \hat{W}(I_1, I_2, I_3), \quad I_1 = \operatorname{tr}\mathbf{b}, \quad I_2 = \tfrac{1}{2}[(I_1)^2 - \operatorname{tr}\mathbf{b}^2], \quad I_3 = \det\mathbf{b} = J^2.$$

The Cauchy stress then follows:

$$\boldsymbol{\sigma} = \frac{2}{J}\left[\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{b} - \frac{\partial W}{\partial I_2}\mathbf{b}^2 + I_3\frac{\partial W}{\partial I_3}\mathbf{I}\right]$$

7.3. Compressible vs Incompressible Formulations

Incompressible: $J = 1$ (constraint), introduce hydrostatic pressure p as Lagrange multiplier:

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\frac{\partial W}{\partial I_1}\mathbf{b} - 2\frac{\partial W}{\partial I_2}\mathbf{b}^{-1}$$

Compressible: use volumetric-isochoric split $W = W_{\mathrm{vol}}(J) + \bar{W}(\bar{I}_1, \bar{I}_2)$ where $\bar{\mathbf{b}} = J^{-2/3}\mathbf{b}$ (unimodular part).

7.4. Neo-Hookean Model

The simplest physically motivated hyperelastic model:

$$W = \frac{\mu}{2}(I_1 - 3) - \mu \ln J + \frac{\lambda}{2}(\ln J)^2$$

where μ is the shear modulus and λ the first Lamé constant.

Cauchy stress:

$$\boldsymbol{\sigma} = \frac{\mu}{J}(\mathbf{b} - \mathbf{I}) + \frac{\lambda \ln J}{J}\mathbf{I}$$

Recovers linear elasticity at small strains.

7.5. Mooney-Rivlin Model

Two-parameter model for rubber-like materials:

$$W = C_1(I_1 - 3) + C_2(I_2 - 3)$$

(incompressible form; add $W_{\text{vol}}(J)$ for compressibility)

- $C_1 = \mu/2$ in the Neo-Hookean limit ($C_2 \to 0$)
- Better captures shear stiffening at moderate strains
- Experimental calibration: shear vs tension data

7.6. Ogden Model

General model written in terms of principal stretches $\lambda_1, \lambda_2, \lambda_3$:

$$W = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p} \left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right)$$

Conditions: $\sum_p \mu_p \alpha_p > 0$ (positive initial shear modulus).

With $N = 3$ terms, fits vulcanized rubber data up to 700% extension.

7.7. Numerical Stress Computation

Algorithm for a given $\mathbf{F}$:

```
def neo_hookean_stress(F, mu, lam):
    """Return Cauchy stress for compressible Neo-Hookean."""
    import numpy as np
    J = np.linalg.det(F)
    b = F @ F.T
    sigma = (mu / J) * (b - np.eye(3)) + (lam * np.log(J) / J) * np.eye(3)
    return sigma
```

7.8. Spatial Elasticity Tensor

Required by FEM for the consistent tangent. For the Neo-Hookean model:

$$= \frac{\lambda}{J}\mathbf{I} \otimes \mathbf{I} + \frac{1}{J}\left(\mu - \lambda \ln J\right)\left(\mathbf{I}\overline{\otimes}\mathbf{I} + \mathbf{I}\underline{\otimes}\mathbf{I}\right)$$

where $(\mathbf{A}\overline{\otimes}\mathbf{B})_{ijkl} = A_{ik}B_{jl}$ and $(\mathbf{A}\underline{\otimes}\mathbf{B})_{ijkl} = A_{il}B_{jk}$.

7.9. Treloar Data and Model Validation

The classic rubber dataset (Treloar 1944) tests models under combined loading:

- Uniaxial extension
- Equi-biaxial extension
- Pure shear

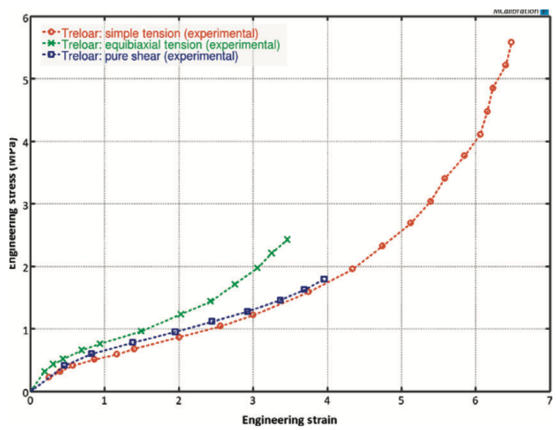


Figure 7.1.: Treloar data fit comparison

7.10. Rubber Elasticity: Molecular Origin

Definition: The ability of elastomers (polymeric materials) to undergo very large, reversible deformations under stress, returning quickly to their original shape upon unloading.

Key Characteristics: - Large strains (often several hundred percent) - Non-linear stress-strain behavior - **Entropic** in origin (unlike metallic elasticity which is enthalpic) - Low Young's modulus, high Poisson's ratio ($\nu \approx 0.5$, nearly incompressible)

Temperature Dependence: Stiffness increases with temperature (consistent with entropic elasticity): $G \propto T$.

7.11. Polymer Network Structure

Elastomers consist of long, flexible polymer chains linked together at specific crosslink points to form a 3D network.

Mechanism: - **Unstressed State:** Chains adopt random coil configurations (high entropy) - **Stretched State:** Chains are forced into more elongated, ordered configurations (low entropy). The network exerts a retractive force trying to return to the high-entropy state.

For elastomers, free energy is dominated by entropy:

$$\Psi = \Psi_{\text{internal}} + \Psi_{\text{entropy}}, \quad \Psi_{\text{entropy}} = -\theta\eta$$

where η is entropy density. Stress arises from the tendency of chains to return to their random coil (highest entropy) state.

7.12. Incompressibility in Hyperelasticity

For rubber-like materials that are nearly incompressible, we use a volumetric-isochoric decomposition:

$$W = W_{\text{iso}}(\bar{I}_1, \bar{I}_2) + W_{\text{vol}}(J)$$

where: - $\bar{I}_1 = I_1 J^{-2/3}$, $\bar{I}_2 = I_2 J^{-4/3}$ are the unimodular (volumetric-free) invariants - $J = \det \mathbf{F} = \sqrt{I_3}$ is the volume ratio - W_{iso} captures shape change (deviatoric response) - W_{vol} captures volume change

For incompressible materials ($J = 1$), we enforce this as a constraint, introducing hydrostatic pressure p as a Lagrange multiplier:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}_{\text{dev}}$$

where the deviatoric stress is derived from W_{iso}.

7.13. Saint Venant-Kirchhoff Model

A simple linear model for the strain energy density:

$$W = \frac{1}{2}\mathbf{E} : \mathbb{D} : \mathbf{E}$$

where $\mathbb{D}$ is defined using Lamé parameters:

$$\mathbb{D} = \lambda\mathbf{I} \otimes \mathbf{I} + 2\mu\mathbb{I}^4$$

with:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

Limitations: Valid only for small to moderate strains, despite using large-strain kinematics (Green-Lagrange strain).

Advantages: Can handle large rotations while remaining simple.

The second Piola-Kirchhoff stress is:

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = 2\frac{\partial W}{\partial \mathbf{C}}$$

7.14. Relationship Between Stress Derivatives

Since $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, we have:

$$d\mathbf{E} = \frac{1}{2} d\mathbf{C}$$

The strain energy can be expressed in either form:

$$d\Psi = \frac{\partial \Psi}{\partial \mathbf{E}} : d\mathbf{E} = \frac{\partial \Psi}{\partial \mathbf{C}} : d\mathbf{C}$$

Substituting $d\mathbf{E} = \frac{1}{2} d\mathbf{C}$:

$$\frac{\partial \Psi}{\partial \mathbf{E}} : \frac{1}{2} d\mathbf{C} = \frac{\partial \Psi}{\partial \mathbf{C}} : d\mathbf{C}$$

Since this holds for arbitrary $d\mathbf{C}$:

$$\mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{E}} = 2 \frac{\partial \Psi}{\partial \mathbf{C}}$$

7.15. Special Deformation Modes

For analyzing hyperelastic models, we often examine specialized deformations:

Mode	Principal stretches	Definition
Uniaxial tension	$\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = \lambda^{-1/2}$	Extension in one direction (incompressible)
Equibiaxial tension	$\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = \lambda^{-2}$	Equal extension in two directions

Mode	Principal stretches	Definition
Pure shear	$\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}$, $\lambda_3 = 1$	Shear without volume change
Simple shear	$\mathbf{F} = \mathbf{I} + \gamma \mathbf{e}_1 \otimes \mathbf{e}_2$	γ is shear strain

These modes are used for model parameter fitting and validation against experimental data.

7.16. Gaussian Chain Statistics

A polymer chain can be modeled as an ideal random walk: - n rigid links of length l (total contour length $L = nl$) - End-to-end distance r with probability distribution - For Gaussian statistics (small extensions):

$$P(r) = \left(\frac{3}{2\pi n l^2}\right)^{3/2} \exp\left(-\frac{3r^2}{2nl^2}\right)$$

The **change in entropy** for a network of chains is:

$$\Delta\eta = -\frac{NK\theta}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) = -W_G$$

where N is the number of chains, K is Boltzmann's constant, and θ is absolute temperature.

Limitations: Gaussian statistics assume freely jointed random coils, valid only for small to moderate stretches.

7.17. Non-Gaussian Chain Behavior

Real polymer chains have finite extensibility (maximum stretch λ_m). The force on a single stretched chain follows:

$$f = \frac{NK\theta}{l}\mathbb{L}^{-1}\left(\frac{r}{nl}\right)$$

where $\mathbb{L}^{-1}$ is the **inverse Langevin function**:

$$\mathbb{L}(x) = 3x + \frac{9}{5}x^3 + \frac{297}{175}x^5 + \cdots$$

Key Property: $\mathbb{L}^{-1}(x) \to \infty$ as $x \to 1$, causing the force (and stress) to increase dramatically at high extensions.

This **finite extensibility** produces the characteristic "stiffening" (upturn) in the stress-strain curve at high strains, which Gaussian statistics cannot capture.

7.18. Arruda-Boyce (8-Chain) Model

Developed: Arruda and Boyce (1993) to capture finite extensibility using non-Gaussian statistics while remaining computationally simple.

Core Concept: Model the polymer network as a collection of identical **unit cells**, each containing 8 chains radiating from the center to the corners of a cube.

Deformation Assumption: - The cube vertices deform affinely with the macroscopic deformation - The 8 chains stretch according to the vertex displacements - This captures the average chain stretch in an isotropic network

The effective chain stretch is related to the first invariant:

$$\lambda_{\text{chain}} = \sqrt{\frac{I_1}{3}}$$

Strain Energy Function:

$$W_{\text{AB}} = NK\theta\sqrt{n}\left[\lambda_{\text{chain}}\beta - \sqrt{n}\ln\left(\frac{\sinh\beta}{\beta}\right)\right]$$

where $\beta = \mathbb{L}^{-1}(\lambda_{\text{chain}}/\sqrt{n})$ is related to the chain force.

Material Parameters: - N = number of segments per chain - K = Boltzmann constant - θ = absolute temperature - Often simplified to two parameters: initial shear modulus and limiting stretch

7.19. Key Features of Arruda-Boyce

- **Physical Basis:** Directly incorporates finite chain extensibility
- **Parameters:** Only two interpretable material parameters
- **Accuracy:** Captures the characteristic 'S-shape' stress-strain curve including high-strain upturn
- **Versatility:** Performs well under various deformation modes (uniaxial, biaxial, shear) using the same parameters
- **Implementation:** Relatively straightforward to implement despite physical basis

7.20. Variational Formulation for Hyperelasticity

For a total Lagrangian (material) formulation, the total potential energy is:

$$\Pi(\mathbf{u}) = \int_{\Omega_0} W(\mathbf{E}(\mathbf{u}))d\Omega_0 - \int_{\Omega_0} \mathbf{u}^T\mathbf{f}^b d\Omega_0 - \int_{\Gamma_{0,t}} \mathbf{u}^T\mathbf{t}d\Gamma_0$$

where $\mathbf{f}^b$ is the body force and $\mathbf{t}$ is the prescribed traction.

The **first variation** (Gâteaux derivative in direction $\delta\mathbf{u}$):

$$\delta\Pi = \int_{\Omega_0} \mathbf{S} : \delta\mathbf{E} d\Omega_0 - \int_{\Omega_0} \delta\mathbf{u}^T \mathbf{f}^b d\Omega_0 - \int_{\Gamma_{0,t}} \delta\mathbf{u}^T \mathbf{t} d\Gamma_0$$

The **Green-Lagrange strain variation** (linear in $\delta\mathbf{u}$, quadratic in $\mathbf{u}$):

$$\delta\mathbf{E} = \mathrm{sym}((\nabla_0\mathbf{u})^T \nabla_0 \delta\mathbf{u})$$

7.21. Linearization for Newton-Raphson

To solve the nonlinear system, we linearize around a current solution $\mathbf{u}_i$ by taking the directional derivative in direction $\Delta\mathbf{u}$:

$$\mathcal{L}[\delta\Pi] = \frac{d}{d\tau} \delta\Pi(\mathbf{u}_i + \tau\Delta\mathbf{u})\Big|_{\tau=0}$$

The strain energy term linearizes as:

$$\mathcal{L}\left[\int_{\Omega_0} \mathbf{S} : \delta\mathbf{E} d\Omega_0\right] = \int_{\Omega_0} [\Delta\mathbf{S} : \delta\mathbf{E} + \mathbf{S} : \Delta(\delta\mathbf{E})] d\Omega_0$$

where: - $\Delta\mathbf{S} = \mathbb{D} : \Delta\mathbf{E}$ (material elasticity tensor $\mathbb{D} = \frac{\partial\mathbf{S}}{\partial\mathbf{E}}$) - $\Delta\mathbf{E} = \mathrm{sym}((\nabla_0\mathbf{u})^T \nabla_0 \Delta\mathbf{u})$ (increment of Green-Lagrange strain) - $\Delta(\delta\mathbf{E}) = \mathrm{sym}((\nabla_0\Delta\mathbf{u})^T \nabla_0 \delta\mathbf{u})$ (variation of strain increment)

The Newton-Raphson system at iteration k becomes:

$$\int_{\Omega_0} [\mathbb{D} : \Delta\mathbf{E} : \delta\mathbf{E} + \mathbf{S} : \Delta(\delta\mathbf{E})] d\Omega_0 = -\mathcal{R}_k(\delta\mathbf{u})$$

where $\mathcal{R}_k$ is the residual at iteration k.

7.22. Material Elasticity Tensor

For the second Piola-Kirchhoff stress, the material elasticity tensor is:

$$\mathbb{D}_{ijkl} = 4\frac{\partial^2 W}{\partial C_{ij} \partial C_{kl}}$$

This fourth-order tensor relates stress increments to strain increments:

$$\Delta\mathbf{S} = \mathbb{D} : \Delta\mathbf{E}$$

For the Neo-Hookean model:

$$W = \frac{\mu}{2}(I_1 - 3) - \mu \ln J + \frac{\lambda}{2}(\ln J)^2$$

The material elasticity tensor takes the form:

$$\mathbb{D}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ik}\delta_{jl} + \mathcal{G}_{ijkl}$$

where $\mathcal{G}$ includes additional terms from the volumetric part.

8. L06 — Viscoelasticity

Linear Viscoelastic Models and Numerical Implementation

8.1. Motivation: Rate-Dependent Elastic Behavior

Viscoelastic materials exhibit:

- **Creep:** increasing strain under constant stress
- **Stress relaxation:** decreasing stress under constant strain
- **Hysteresis:** energy dissipation in cyclic loading

Examples: polymers, soft biological tissues, asphalt, filled rubber.

Key distinction from plasticity: fully recoverable in principle (no permanent set), but **time-dependent**.

8.2. The Maxwell Model

Spring (stiffness E) in **series** with dashpot (viscosity η):

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

Relaxation time $\tau = \eta/E$. Under constant strain ε_0:

$$\sigma(t) = E\varepsilon_0\, e^{-t/\tau}$$

→ Captures relaxation, but predicts indefinite creep.

8.3. The Kelvin-Voigt Model

Spring and dashpot in **parallel**:

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

Under constant stress σ_0:

$$\varepsilon(t) = \frac{\sigma_0}{E}\left(1 - e^{-t/\tau}\right)$$

→ Captures creep saturation, but cannot relax instantaneously.

8.4. Standard Linear Solid (SLS)

One spring in parallel with a Maxwell element. Captures both creep and relaxation:

$$\sigma + \tau_\sigma\dot{\sigma} = E_\infty\varepsilon + E_0\tau_\sigma\dot{\varepsilon}$$

Relaxed modulus $E_\infty < E_0$ (instantaneous).

8.5. Generalized Maxwell Model (Prony Series)

Multiple Maxwell branches in parallel:

$$\sigma(t) = \varepsilon_0 \left[E_\infty + \sum_{i=1}^{N} E_i \, e^{-t/\tau_i} \right]$$

- Excellent fit to broad relaxation spectra
- Each branch adds two parameters (E_i, τ_i)
- DMA experiments needed for identification

8.6. Hereditary Integral Representation

The stress history in terms of the **relaxation kernel** $G(t)$:

$$\sigma(t) = \int_0^t G(t-s) \, \dot{\varepsilon}(s) \, ds$$

For the Prony series:

$$G(t) = E_\infty + \sum_{i=1}^{N} E_i \, e^{-t/\tau_i}$$

8.7. Numerical Implementation: Recursive Update

Each Maxwell branch maintains an **internal variable** h_i (history stress). Efficient recursive update over time step Δt:

$$h_i^{n+1} = e^{-\Delta t/\tau_i} \, h_i^n + E_i (1 - e^{-\Delta t/\tau_i}) \, \Delta\varepsilon$$

Total stress:

$$\sigma^{n+1} = E_\infty \varepsilon^{n+1} + \sum_i h_i^{n+1}$$

Algorithmic tangent: $\partial\sigma/\partial\varepsilon = E_\infty + \sum_i E_i(1 - e^{-\Delta t/\tau_i})\Delta t/\tau_i$ (approx).

8.8. 3D Finite Viscoelasticity

For finite deformations, use volumetric-isochoric split of the free energy:

$$\Psi = \Psi_{\mathrm{vol}}(J) + \bar{\Psi}_\infty(\bar{\mathbf{C}}) + \sum_i \Gamma_i(\bar{\mathbf{C}}, \tilde{\mathbf{C}}_i)$$

where $\tilde{\mathbf{C}}_i$ are internal (viscous) deformation variables.

> **Notation — three distinct Cauchy-Green–like tensors in this section.** The symbol $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the usual right Cauchy-Green tensor (see L03). The overbar $\bar{\mathbf{C}} = \bar{\mathbf{F}}^T\bar{\mathbf{F}}$ is the **isochoric (unimodular) part** of $\mathbf{C}$ obtained from the volumetric–isochoric split $\mathbf{F} = J^{1/3}\bar{\mathbf{F}}$ so that $\det\bar{\mathbf{C}} = 1$. The tilde $\tilde{\mathbf{C}}_i$ denotes **internal viscous deformation variables** (one per Maxwell branch i) — they are state variables evolved by the viscoelastic rate equation, not kinematic quantities derived from $\mathbf{F}$. In particular, $\tilde{\mathbf{C}}_i \neq \mathbf{C}$ and $\tilde{\mathbf{C}}_i \neq \bar{\mathbf{C}}$ at a generic instant.

Evolution equation for each internal variable mirrors the 1D recursive formula but in tensor form.

9. L07 — Plasticity: Theory

Yield Criteria, Flow Rules, Hardening, and Loading/Unloading Conditions

9.1. Phenomenology of Plasticity

Characteristic features:

- **Irreversible deformation** upon load removal
- A **yield threshold** — elastic below, plastic above
- **Hardening:** yield stress increases with accumulated plastic strain
- **Bauschinger effect:** kinematic hardening under reversed loading

The **additive strain decomposition** (small strains):

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$$

> **Regime.** Unless stated otherwise, this chapter (and L08, L09) assumes the **small-strain regime**: $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$, with additive decomposition $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$. Finite-strain plasticity uses a multiplicative split $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ (outside scope).

9.2. Deviatoric Stress and Pressure

Hydrostatic pressure p and deviatoric stress $\mathbf{s}$:

$$p = -\tfrac{1}{3}\operatorname{tr}\boldsymbol{\sigma}, \qquad \mathbf{s} = \boldsymbol{\sigma} + p\mathbf{I}.$$

Von Mises equivalent stress:

$$\sigma_{\text{eq}} = \sqrt{\tfrac{3}{2}\mathbf{s} : \mathbf{s}} = \sqrt{3J_2}, \qquad J_2 = \tfrac{1}{2}\mathbf{s} : \mathbf{s}.$$

9.3. Yield Criteria

Von Mises (J2): pressure-independent, metallic materials:

$$f(\boldsymbol{\sigma}, \kappa) = \sigma_{\text{eq}} - \sigma_y(\kappa) \leq 0$$

Tresca: maximum shear stress criterion — inscribes Von Mises in principal-stress space.

Drucker-Prager: pressure-dependent, cohesive-frictional materials:

$$f = \alpha_{\text{DP}} I_1 + \sqrt{J_2} - k \leq 0$$

Here α_{DP} is the Drucker-Prager pressure-sensitivity coefficient — distinct from the backstress $\boldsymbol{\alpha}$ used later in this chapter and from the equivalent plastic strain used elsewhere.

Mohr-Coulomb: $\tau_{\max} = c - \sigma_n \tan\phi$ on the slip plane.

9.4. Flow Rule

The plastic strain rate is governed by a **flow rule**:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\,\mathbf{m}(\boldsymbol{\sigma})$$

(overdot denotes the material time derivative $\dot{(\cdot)} = D(\cdot)/Dt$ — see L03)

Associated (normality): $\mathbf{m} = \partial f/\partial\boldsymbol{\sigma}$ — plastic flow normal to yield surface. Follows from maximum plastic dissipation principle; ensures convexity.

Non-associated: $\mathbf{m} \neq \partial f/\partial\boldsymbol{\sigma}$ — used when dilatancy must be independently controlled (Drucker-Prager, soils).

9.5. Hardening Rules

Isotropic hardening: yield surface expands uniformly.

$$\sigma_y = \sigma_{y0} + H_{\mathrm{iso}}\bar{\varepsilon}^p, \qquad \dot{\bar{\varepsilon}}^p = \sqrt{\tfrac{2}{3}\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p}$$

Kinematic hardening: yield surface translates — captures Bauschinger effect.

$$f = \sigma_{\mathrm{eq}}(\boldsymbol{\sigma} - \boldsymbol{\alpha}) - \sigma_{y0} \leq 0, \qquad \dot{\boldsymbol{\alpha}} = \tfrac{2}{3}H_{\mathrm{kin}}\,\dot{\boldsymbol{\varepsilon}}^p$$

Here $\boldsymbol{\alpha}$ is the **backstress tensor** (deviatoric in J2; centre of the yield surface in deviatoric-stress space) and H_{kin} is the **linear kinematic (Prager) hardening modulus**. Same symbols carry through L08 and L08-appendix.

Mixed (combined) hardening: both effects simultaneously.

9.6. Kuhn-Tucker Loading/Unloading Conditions

Yield function sign convention. Throughout L07–L09, $f \leq 0$ is the elastic (admissible) region, $f = 0$ is the yield surface, and $f > 0$ is inadmissible (non-physical — the return-mapping algorithms of L08 project back to $f = 0$).

The standard **Karush-Kuhn-Tucker (KKT) conditions** govern plastic flow:

$$f \leq 0, \qquad \dot{\gamma} \geq 0, \qquad \dot{\gamma}\, f = 0.$$

Consistency condition (during plastic loading, $f = 0$ and $\dot{f} = 0$):

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \kappa}\dot{\kappa} = 0$$

This determines $\dot{\gamma}$.

9.7. Elasto-Plastic Tangent Modulus

The **continuum** elasto-plastic modulus $\mathbb{C}^{ep}$ relates stress and strain rates in the plastic loading regime:

$$\dot{\boldsymbol{\sigma}} = \mathbb{C}^{ep} : \dot{\boldsymbol{\varepsilon}}, \qquad \mathbb{C}^{ep} = \mathbb{C}^{e} - \frac{(\mathbb{C}^{e} : \mathbf{n}) \otimes (\mathbf{n} : \mathbb{C}^{e})}{h + \mathbf{n} : \mathbb{C}^{e} : \mathbf{n}}$$

where $\mathbf{n} = \partial f / \partial \boldsymbol{\sigma}$ and h is the hardening modulus.

Note: $\mathbb{C}^{ep}$ is symmetric for associated flow; non-symmetric for non-associated.

9.8. Material Stability

Drucker's stability postulate: for stable materials, the plastic work increment is non-negative:

$$d\boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^p \geq 0$$

Consequences:

- Yield surface must be **convex**
- Flow must obey **normality** (associated rule)
- $\mathbb{C}^{ep}$ is positive semi-definite

Loss of stability $\rightarrow$ localization (shear bands), which requires special regularization.

9.9. One-Dimensional Elastoplasticity

9.9.1. Fundamentals

Small strain additive decomposition:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$$

Stress is related to elastic strain only. Plastic strain is an **internal variable** evolving with plastic deformation.

9.9.2. Elastic predictor, plastic corrector procedure

Given trial (elastic) stress and current state variables:

$$\sigma^{\mathrm{tr}} = \sigma^n + E\Delta\varepsilon, \quad \text{Yield check:} \quad f^{\mathrm{tr}} = |\sigma^{\mathrm{tr}}| - \sigma_y^n$$

If elastic ($f^{\mathrm{tr}} \leq 0$): Accept trial; no plastic flow.

If plastic ($f^{\mathrm{tr}} > 0$): Return to yield surface via plastic consistency:

$$|\sigma^{\mathrm{tr}} - \mathrm{sgn}(\sigma^{\mathrm{tr}})E\Delta\varepsilon_p| = \sigma_y^n + H_{\mathrm{iso}}\Delta\varepsilon_p$$

Solving: $\Delta\varepsilon_p = \frac{f^{\mathrm{tr}}}{E+H_{\mathrm{iso}}}$ and $\sigma^{n+1} = \sigma^{\mathrm{tr}} - \mathrm{sgn}(\sigma^{\mathrm{tr}})E\Delta\varepsilon_p$.

9.9.3. Isotropic hardening

Yield stress evolves with accumulated plastic strain:

$$\sigma_y = \sigma_{y0} + H_{\mathrm{iso}}\bar{\varepsilon}^p$$

The plastic modulus H_{iso} relates stress increment to plastic strain increment:

$$H_{\mathrm{iso}} = \frac{EE_t}{E - E_t}, \quad \text{where } E_t = \frac{EH_{\mathrm{iso}}}{E + H_{\mathrm{iso}}}$$

9.9.4. Kinematic hardening

Yield surface translates (back stress α evolves):

$$\text{Effective stress:} \quad \eta = \sigma - \alpha, \quad \text{Yield:} \quad |\eta| = \sigma_{y0}$$

Back stress evolution: $\alpha^{n+1} = \alpha^n + \mathrm{sgn}(\eta)H_{\mathrm{kin}}\Delta\varepsilon_p$.

Captures **Bauschinger effect** (yield strength reduction upon load reversal).

9.9.5. Combined hardening

Mix isotropic and kinematic via parameter $\beta \in [0, 1]$:

$$\sigma_y^{n+1} = \sigma_y^n + (1-\beta)H\Delta\varepsilon_p, \quad \alpha^{n+1} = \alpha^n + \beta H\Delta\varepsilon_p$$

In this combined-hardening simplification, H is a single mixed modulus; compare L08-appendix where iso and kin moduli are split as H_{iso} and H_{kin}.

$\beta = 0 \to$ isotropic; $\beta = 1 \to$ kinematic.

9.10. Multi-Dimensional Theory

9.10.1. Deviatoric stress and strain invariants

Hydrostatic component and deviator:

$$\boldsymbol{\sigma}_m = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}, \quad \mathbf{s} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_m$$

Second invariant of deviatoric stress:

$$J_2 = \tfrac{1}{2}\mathbf{s} : \mathbf{s}$$

9.10.2. Von Mises yield criterion (J2 plasticity)

Pressure-insensitive (applies to metals):

$$f = \sigma_{\text{eq}} - \sigma_y(\bar{\varepsilon}^p) = \sqrt{3J_2} - \sigma_y = 0$$

Where $\sigma_{\text{eq}} = \sqrt{\tfrac{3}{2}\mathbf{s} : \mathbf{s}}$ is the **equivalent (von Mises) stress**.

Plastic flow is deviatoric (dilatation elastic, distortion plastic).

9.10.3. Effective plastic strain

Conjugate measure to equivalent stress:

$$\bar{\varepsilon}^p = \int_0^t \dot{\bar{\varepsilon}}^p \, d\tau, \quad \dot{\bar{\varepsilon}}^p = \sqrt{\tfrac{2}{3}\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p}$$

Plasticity relations in principal stress space often cleaner; spectral decomposition aligns principal stresses with flow direction.

9.10.4. Normality and plastic flow

For associated plasticity (most common):

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\gamma}\frac{3}{2}\frac{\mathbf{s}}{\sigma_{\mathrm{eq}}}$$

Direction normal to yield surface; ensures **convexity and stability**.

9.10.5. Multi-axial hardening models

Isotropic: yield surface expands uniformly in deviatoric space.

Kinematic: surface translates (back stress tensor $\boldsymbol{\alpha}$ evolves).

$$f = \|\mathbf{s} - \boldsymbol{\alpha}\| - \sqrt{\tfrac{2}{3}}\sigma_{y0}$$

Combined: both mechanisms active simultaneously.

9.10.6. Pressure-dependent criteria

Drucker-Prager:

$$f = \alpha_{\mathrm{DP}} I_1 + \sqrt{J_2} - k \leq 0$$

Common in geomechanics and concrete; $I_1 = \mathrm{tr}(\boldsymbol{\sigma})$ is the first invariant (hydrostatic stress).

Mohr-Coulomb: defined by friction angle ϕ and cohesion c; forms a hexagonal cone in principal stress space.

Non-associated flow rule often needed to control dilatancy.

9.11. Finite Deformations and Objectivity

9.11.1. Objective stress rates

Constitutive laws written in rate form must use **objective rates** to be frame-indifferent:

$$\dot{\boldsymbol{\sigma}}^{\mathcal{J}} = \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w}$$

Jaumann rate: uses spin tensor $\mathbf{w} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)$ (skew part of velocity gradient).

9.11.2. Finite rotation: midpoint configuration

For finite rotations, use intermediate (midpoint) configuration to avoid spurious stress rotation:

Rotate stress to midpoint configuration → perform plasticity update → rotate back:

$$\bar{\boldsymbol{\sigma}} = \mathbf{R}\boldsymbol{\sigma}^n\mathbf{R}^T, \quad \text{update with } \bar{\boldsymbol{\sigma}}, \quad \boldsymbol{\sigma}^{n+1} = \mathbf{R}^T\bar{\boldsymbol{\sigma}}^{n+1}\mathbf{R}$$

R extracted from incremental deformation gradient via polar decomposition or exponential map.

9.11.3. Multiplicative decomposition (large strains)

Deformation gradient splits elastically and plastically:

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$$

Plastic part $\mathbf{F}^p$ lies in an unobservable **intermediate configuration** (stress-free after plastic flow).

Enables consistent treatment of large elastic *and* large plastic strains via hyperelasticity + plasticity in principal stress space.

> **Worked examples:** fully numerical derivations of the RRM for combined Prager + power-law isotropic hardening are in L08 Appendix — Worked Examples.

10. L08 — Plasticity: Algorithms

Return Mapping, Consistent Tangent, and Rate-Dependent Extensions

Regime. This chapter assumes **small-strain plasticity**: $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ with additive decomposition $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$.

10.1. Incremental Elasto-Plasticity

Over a time step $[t_n, t_{n+1}]$ with strain increment $\Delta\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_n + \Delta\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon}^e_{n+1} = \boldsymbol{\varepsilon}^e_n + \Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^p.$$

Elastic predictor — plastic corrector paradigm:

1. *Freeze* plastic strain: assume $\Delta\boldsymbol{\varepsilon}^p = \mathbf{0}$.
2. Compute trial stress: $\boldsymbol{\sigma}^{\mathrm{tr}} = \boldsymbol{\sigma}_n + \mathbb{C}^e : \Delta\boldsymbol{\varepsilon}$.

where $f^{\mathrm{tr}} := f(\boldsymbol{\sigma}^{\mathrm{tr}}, q_n)$ is the trial yield function value.

3. Check yield: if $f^{\mathrm{tr}} \leq 0 \rightarrow$ **elastic step**, accept trial.
4. If $f^{\mathrm{tr}} > 0 \rightarrow$ **plastic step**, correct by return mapping.

10.2. Radial Return Algorithm (J2)

For J2 plasticity the return mapping has a **closed-form** solution (radial return in deviatoric space):

Given $\mathbf{s}^{\text{tr}}$ (trial deviatoric stress) and $\bar{\varepsilon}^p_n$:

$$\Delta\gamma = \frac{\sigma^{\text{tr}}_{\text{eq}} - \sigma_{y0} - H\bar{\varepsilon}^p_n}{2\mu + \frac{2}{3}H} \cdot \frac{1}{1} > 0$$

Updated variables:

$$\mathbf{s}_{n+1} = \left(1 - \frac{3\mu\Delta\gamma}{\sigma^{\text{tr}}_{\text{eq}}}\right)\mathbf{s}^{\text{tr}}, \qquad \bar{\varepsilon}^p_{n+1} = \bar{\varepsilon}^p_n + \Delta\gamma.$$

```python
def j2_return_mapping(s_tr, ep_bar_n, mu, sigma_y0, H):
    """Radial return for J2 plasticity with linear isotropic hardening."""
    import numpy as np
    seq_tr = np.sqrt(1.5 * np.tensordot(s_tr, s_tr))
    f_tr = seq_tr - (sigma_y0 + H * ep_bar_n)
    if f_tr <= 0.0:                                # elastic step
        return s_tr, ep_bar_n
    # plastic step
    dgamma = f_tr / (2 * mu + 2/3 * H)
    s_new  = (1 - 3 * mu * dgamma / seq_tr) * s_tr
    ep_bar_new = ep_bar_n + dgamma
    return s_new, ep_bar_new
```

10.3. Consistent (Algorithmic) Tangent

The **continuum** tangent $\mathbb{C}^{ep}$ is **not** the right modulus for FEM — it loses quadratic convergence. The **consistent (algorithmic) tangent** $\mathbb{C}^{\text{alg}} = \frac{d\boldsymbol{\sigma}_{n+1}}{d\boldsymbol{\varepsilon}_{n+1}}$ is required.

For J2 with radial return:

$$\mathbb{C}^{\text{alg}} = \kappa\,\mathbf{I}\otimes\mathbf{I} + 2\mu\theta_1\,\mathbb{P}_{\text{dev}} - 2\mu\theta_2\,\mathbf{n}\otimes\mathbf{n}$$

where $\mathbf{n} = \mathbf{s}^{\text{tr}}/\|\mathbf{s}^{\text{tr}}\|$, $\theta_1 = 1 - 3\mu\Delta\gamma/\sigma_{\text{eq}}^{\text{tr}}$, $\theta_2 = 1/(1 + H/3\mu) - (1 - \theta_1)$.

10.4. Cut-Plane (Closest-Point Projection) Algorithm

General return mapping for **any yield surface** (not only J2):

Minimise $\frac{1}{2}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\text{tr}}\|^2_{\mathbb{C}^{e-1}}$ subject to $f(\boldsymbol{\sigma}) = 0$.

Iterative Newton loop:

$$\begin{bmatrix}\boldsymbol{\sigma}\\ \Delta\gamma\end{bmatrix}^{k+1} = \begin{bmatrix}\boldsymbol{\sigma}\\ \Delta\gamma\end{bmatrix}^{k} - \mathbf{J}^{-1}\begin{bmatrix}\mathbf{r}_\sigma\\ f\end{bmatrix}^{k}$$

where $\mathbf{J}$ is the Jacobian of the residual system.

10.5. Rate-Dependent Plasticity (Viscoplasticity)

Perzyna-type: allows stress to lie outside the yield surface — the yield function serves as an overstress measure:

$$\dot{\bar{\varepsilon}}^p = \frac{1}{\eta}\langle f/\sigma_y\rangle^N, \qquad \langle x\rangle = \max(0, x).$$

Consistency viscoplasticity (Wang et al.): extends standard KKT conditions to rate-dependent case with a viscoplastic yield surface $f^{vp}(\boldsymbol{\sigma}, \kappa, \dot{\kappa}) = 0$.

Update algorithm: implicit scheme, Newton solve for $\Delta\gamma$.

10.6. Pressure-Dependent Plasticity: Drucker-Prager

Drucker-Prager yield function:

$$f = \alpha I_1 + \sqrt{J_2} - k \le 0$$

Return mapping: no closed form → cut-plane or linearized iterations.

Special cases: apex return (if trial stress maps to the apex cone).

10.7. Pressure-Dependent Plasticity: Mohr-Coulomb

Mohr-Coulomb in principal-stress space (Koiter's multi-surface plasticity):

$$f_i = \sigma_j - \sigma_k - (\sigma_j + \sigma_k)\sin\phi - 2c\cos\phi \le 0, \quad i, j, k = 1, 2, 3$$

Six yield planes form the hexagonal pyramid.

Return mapping: depends on which face(s) are active — requires case distinction (edge/apex/face returns).

10.8. Multi-Dimensional Return Mapping: Implicit Integration

General approach for **any yield surface** using **backward Euler** (fully implicit):

Newton-Raphson system:

$$\boldsymbol{\sigma}_{n+1} = \mathbf{C}^e : (\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n^p - \Delta\gamma \mathbf{n}_{n+1}), \quad f(\boldsymbol{\sigma}_{n+1}, q_{n+1}) = 0,$$

where $\Delta\gamma$ and state variables are unknowns. Iterations via:

$$\mathbf{J}\begin{bmatrix}\Delta\boldsymbol{\sigma}\\ \Delta\Delta\gamma\end{bmatrix}^{(k)} = -\begin{bmatrix}\text{stress residual}\\ f\end{bmatrix}^{(k)}$$

Jacobian combines elasticity, yield surface curvature, and hardening effects.

10.9. Consistent Tangent: Multi-Dimensional

The **consistent (algorithmic) tangent** for general plasticity:

$$\mathbb{C}^{\text{alg}} = \mathbb{C}^e - \frac{\mathbb{C}^e : \mathbf{n} \otimes \mathbf{n} : \mathbb{C}^e}{h + \mathbf{n} : \mathbb{C}^e : \mathbf{n}}$$

where: - $\mathbf{n} = \partial f/\partial\boldsymbol{\sigma}$ (flow direction)[1] - $h = -\partial f/\partial q \cdot dq/d\bar{\varepsilon}^p$ (hardening modulus, negative for softening)

Critical: Use $\mathbb{C}^{\text{alg}}$ for FEM assembly. Continuum $\mathbb{C}^{ep}$ loses quadratic convergence.

10.10. Apex and Edge Returns

Drucker-Prager apex: if trial stress maps "inside" the cone apex, return directly to apex (constrain deviatoric flow).

Mohr-Coulomb edges: hexagonal pyramid has 12 edges. Returning to an edge means active constraints on two yield planes. Requires 2D projection in $(I_1, \sqrt{J_2})$ space.

[1] For **associated** flow rules, the flow direction equals the yield-normal: $\mathbf{n} = \partial f/\partial\boldsymbol{\sigma}$. For **non-associated** flow rules, a separate plastic potential $g(\boldsymbol{\sigma})$ defines the direction $\mathbf{n} = \partial g/\partial\boldsymbol{\sigma}$ (see L07 for the general setup). Unless stated otherwise, this chapter assumes associated flow ($g = f$).

Robust strategy: Check all candidate faces/edges; select the closest admissible surface.

10.11. Viscoplasticity: Rate-Dependent Plasticity

Perzyna model: allows stress overstress beyond yield surface:

$$\dot{\bar{\varepsilon}}^p = \frac{1}{\eta}\left\langle \frac{f}{\sigma_y} \right\rangle^N, \quad \langle x \rangle = \max(0, x)$$

Overstress relaxes via viscous flow; material "creeps" above yield.

Penalty-type viscoplasticity: embedded directly in time-stepping (implicit method can use smaller steps for rate effects).

Key parameters: η (viscosity) and N (power law exponent).

10.12. Incremental Form: General Hardening

For combined isotropic-kinematic hardening in incremental form:

$$\Delta\boldsymbol{\sigma} = \mathbb{C}^e : \Delta\boldsymbol{\varepsilon}^e = \mathbb{C}^e : (\Delta\boldsymbol{\varepsilon} - \Delta\gamma\mathbf{n})$$

Yield condition at step $n+1$:

$$f = \|\mathbf{s}_{n+1} - \boldsymbol{\alpha}_{n+1}\| - \sqrt{\frac{2}{3}}[\sigma_{y0} + (1-\beta)H\bar{\varepsilon}^p_{n+1}] = 0$$

Back stress evolves:

$$\boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_n + \frac{2}{3}\beta H\Delta\gamma\mathbf{n}$$

Effective plastic strain:

$$\bar{\varepsilon}^p_{n+1} = \bar{\varepsilon}^p_n + \sqrt{\frac{2}{3}}\Delta\gamma$$

For a fully worked example combining linear kinematic (Prager) and **power-law** isotropic hardening — including the non-linear scalar residual $R(\Delta\gamma) = 0$, its local Newton solve, and the consistent-tangent outline — see L08 Appendix — Worked Examples.

10.13. Consistent Tangent: J2 Plasticity Explicit Form

For J2 with linear hardening, the consistent tangent has explicit form:

$$\mathbb{C}^{\text{alg}} = \kappa(\mathbf{I} \otimes \mathbf{I}) + 2\mu\left[\theta_1 \mathbb{P}_{\text{dev}} - \theta_2 \mathbf{n} \otimes \mathbf{n}\right]$$

where: - $\mathbb{P}_{\text{dev}} = \mathbb{I} - \frac{1}{3}(\mathbf{I} \otimes \mathbf{I})$ is deviatoric projector - $\theta_1 = 1 - 3\mu\Delta\gamma/\sigma^{\text{tr}}_{\text{eq}}$ captures deviatoric scaling - $\theta_2 = 1 - \theta_1 + H/(3\mu)$ balances elastic vs. plastic stiffness

Ensures quadratic N-R convergence in FEM.

10.14. Implementation Notes

Voigt convention. Throughout this chapter, the 4th-order elasticity tensor $\mathbb{C}^e$ is stored as a 6×6 matrix using **Voigt ordering**: index pairs `(i,j)` map to a single index as `[1,1]→1`, `[2,2]→2`, `[3,3]→3`, `[2,3]=[3,2]→4`, `[1,3]=[3,1]→5`, `[1,2]=[2,1]→6`. Stress and strain vectors are stored with the same ordering. Note that strain

> components 4–6 are engineering shears ($\gamma_{ij} = 2\varepsilon_{ij}$ for $i \neq j$), so Voigt is not an isometry — use **Mandel ordering** (scaled by $\sqrt{2}$) if isometric inner products are required.

State variables to store at each Gauss point: - Cauchy stress $\boldsymbol{\sigma}$ - Effective plastic strain $\bar{\varepsilon}^p$ (or cumulative plastic multiplier $\sum \Delta\gamma$) - Back stress $\boldsymbol{\alpha}$ (if kinematic hardening) - Accumulated damage D (if damage is coupled)

Algorithm summary: 1. Compute trial stress from elastic predictor 2. Check yield; if elastic, update stresses only 3. If plastic: implicit return mapping (Newton loop) → updated $\boldsymbol{\sigma}, \bar{\varepsilon}^p, \boldsymbol{\alpha}, q$ 4. Compute algorithmic tangent for FEM assembly 5. Assemble element stiffness and internal force; proceed to next global iteration

Common pitfalls: - Using continuum tangent instead of algorithmic → poor convergence - Stress drift from yield surface → use consistent/accurate return mapping - Integration error accumulation → monitor time step size

> **Worked example:** L08 Appendix — Worked Examples carries out the full RRM derivation for the combined linear kinematic (Prager) + power-law isotropic hardening case, including the non-linear scalar residual $R(\Delta\gamma) = 0$ and its local Newton solve.

11. L09 — Damage Mechanics

Continuum Damage, Fracture, and Coupled Problems

This chapter covers two coupled mechanics frameworks. (1) **Continuum Damage Mechanics (CDM)** — introduces a scalar damage variable $D \in [0, 1]$ that degrades elastic stiffness. (2) **Pressure-dependent plasticity** — Drucker-Prager and Mohr-Coulomb yield criteria that incorporate hydrostatic stress via I_1 or p. The two can be combined in coupled damage-plasticity models.

Regime. Unless stated otherwise, this chapter (and L08, L09) assumes the **small-strain regime**: $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$, with additive decomposition $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$. Finite-strain plasticity uses a multiplicative split $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$ (outside scope).

11.1. Motivation: Degradation of Stiffness

Unlike plasticity (permanent strain), **damage** describes the progressive degradation of material stiffness due to microcrack growth and void coalescence.

Observable signatures:

- Reduction of the elastic modulus
- **Stress softening** (load-bearing capacity decreases with deformation)
- Final fracture / failure

Continuum Damage Mechanics (CDM) describes this via **internal damage variables** without resolving individual cracks.

11.2. Isotropic Damage Model

A scalar damage variable $D \in [0, 1]$:

- $D = 0$: virgin (undamaged) material
- $D = 1$: fully damaged (no load-carrying capacity)

Effective stress concept (Lemaitre):

$$\tilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{1 - D}$$

Constitutive law with damage:

$$\boldsymbol{\sigma} = (1 - D)\, \mathbb{C}^e : \boldsymbol{\varepsilon}$$

11.3. Strain Equivalence Principle

Strain equivalence (Lemaitre): the strain in the damaged material under nominal stress $\boldsymbol{\sigma}$ equals the strain in the undamaged material under effective stress $\tilde{\boldsymbol{\sigma}}$.

This allows re-use of undamaged material models by simply replacing $\boldsymbol{\sigma} \to \tilde{\boldsymbol{\sigma}}$.

Energy equivalence (Cordebois & Sidoroff): alternative that gives different predictions for shear.

11.4. Damage Evolution Law

Thermodynamic driving force (energy release rate Y):

$$Y = -\frac{\partial \Psi}{\partial D} = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}$$

Damage criterion and evolution:

$$g(Y, D) = Y - r(D) \le 0, \qquad \dot{D} = \dot{\mu}\frac{\partial g}{\partial Y},$$

where $r(D)$ is the damage threshold function (analogous to yield stress hardening).

11.5. Coupling Damage with Plasticity

Combined elasto-plastic-damage model free energy:

$$\Psi = (1 - D)\hat{\Psi}^e(\boldsymbol{\varepsilon}^e) + \Psi^p(\bar{\varepsilon}^p)$$

The stress:

$$\boldsymbol{\sigma} = (1 - D)\mathbb{C}^e : \boldsymbol{\varepsilon}^e$$

Both yield surface and damage criterion must be checked at each step. Order of priority depends on the coupling model chosen.

11.6. Softening and Regularization

Softening (negative tangent modulus) triggers:

- Loss of ellipticity of governing equations
- **Mesh-dependent** FEM solutions (localization into a band of one element width)

Regularization methods:

- **Nonlocal damage:** replace Y by a weighted spatial average $\bar{Y}$
- **Gradient damage:** add $-c\,\Delta D$ to the damage driving force
- **Phase-field fracture:** variational approach with intrinsic length scale

11.7. Phase-Field Approach to Fracture

Introduce a smooth phase-field $\phi \in [0,1]$ interpolating between intact ($\phi = 0$) and fractured ($\phi = 1$):

$$\mathcal{E}(\mathbf{u}, \phi) = \int_\Omega \left[g(\phi)\Psi^e_+(\boldsymbol{\varepsilon}) + \Psi^e_-(\boldsymbol{\varepsilon}) + \frac{G_c}{2}\left(\frac{\phi^2}{l_0} + l_0|\nabla\phi|^2\right)\right] dV$$

where $g(\phi) = (1-\phi)^2$ is the degradation function and l_0 the regularization length.

Advantages: no explicit crack tracking, natural branching/merging.

11.8. Thermo-Mechanical Coupling

Temperature θ affects yield stress and elastic moduli:

$$\sigma_y = \sigma_y(\bar{\varepsilon}^p, \theta), \qquad \boldsymbol{\varepsilon}^{\text{th}} = \alpha(\theta - \theta_{\text{ref}})\mathbf{I}.$$

Energy equation with heat generated by plastic dissipation (Taylor-Quinney coefficient $\beta \approx 0.9$):

$$\rho c_p \dot{\theta} = \beta\, \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \nabla \cdot (k \nabla \theta)$$

Strong coupling requires staggered or monolithic FEM solvers.

11.9. Pressure-Dependent Yield Criteria (Recap)

Materials like soils, rocks, and concrete exhibit plastic behavior that **depends strongly on hydrostatic pressure**, unlike metals (which follow von Mises, pressure-independent).

Why Pressure Dependence Matters: - Frictional effects: higher normal stress increases shear strength - Dilatational effects: volume changes coupled to plastic flow - Confinement: material becomes stronger under compression

All pressure-dependent models consist of: 1. **Additive strain decomposition:** $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$ 2. **Hypoelastic law:** $\dot{\boldsymbol{\sigma}} = \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}}^e$ 3. **Yield function:** $f(\boldsymbol{\sigma}, q) \leq 0$ (depends on hydrostatic pressure p) 4. **Flow rule:** $\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma} \frac{\partial g(\boldsymbol{\sigma}, q)}{\partial \boldsymbol{\sigma}}$ (often non-associative: $g \neq f$) 5. **Hardening law:** $\dot{q} = \dot{\gamma} h(\boldsymbol{\sigma}, q)$ (e.g., cohesion increases with plastic strain) 6. **Kuhn-Tucker conditions:** $\dot{\gamma} \geq 0$, $f \leq 0$, $\dot{\gamma} f = 0$

11.10. Mohr-Coulomb Criterion

The **Mohr-Coulomb criterion** is the classical model for frictional materials (soils, rocks, concrete).

Physical Interpretation: Yielding occurs when shear stress on any plane reaches:

$$\tau = c - \sigma_n \tan\phi$$

where: - τ = shear stress on the failing plane - σ_n = normal stress on the failing plane - c = cohesion (material strength when $\sigma_n = 0$) - ϕ = internal friction angle

This gives a linear failure envelope in (σ_n, τ) space.

Yield Function in Terms of Principal Stresses:

For principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$:

$$\Phi(\boldsymbol{\sigma}, c) = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi = 0$$

Using invariants (with p = mean pressure, θ = Lode angle, J_2, J_3 = stress invariants):

$$\Phi = \left(\cos\theta - \frac{\sin\theta\sin\phi}{\sqrt{3}}\right)\sqrt{J_2} + p\sin\phi - c\cos\phi = 0$$

Key Properties: - Non-smooth yield surface: hexagonal pyramid in principal stress space - Apex at $p = c\cot\phi$ (tensile limit) - Generally **non-associative flow rule** (dilation angle $\psi \neq \phi$) - **Hardening:** cohesion becomes a function of accumulated plastic strain: $c = c(\bar{\varepsilon}^p)$

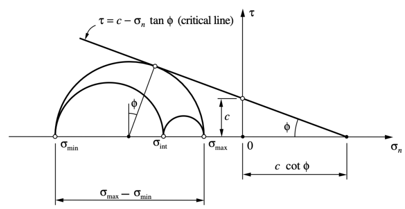


Figure 11.1.: Mohr-Coulomb surface

11.11. Tresca Criterion

The **Tresca yield criterion** (pressure-insensitive version of M-C) states that plasticity begins when the maximum shear stress reaches a critical value:

$$\tau_{\max} = \frac{1}{2}(\sigma_{\max} - \sigma_{\min})$$

Yield Function:

$$\Phi(\boldsymbol{\sigma}) = (\sigma_1 - \sigma_3) - \sigma_y(\bar{\varepsilon}^p) = 0$$

where $\sigma_y(\bar{\varepsilon}^p)$ is the uniaxial yield stress (function of hardening variable $\bar{\varepsilon}^p$).

Yield Surface: A hexagonal prism in principal stress space (axis = hydrostatic line).

Multisurface Representation: Six yield surfaces based on principal stress pairs:

$$\Phi_1 = \sigma_1 - \sigma_3 - \sigma_y, \quad \Phi_2 = \sigma_2 - \sigma_3 - \sigma_y, \quad \Phi_3 = \sigma_1 - \sigma_2 - \sigma_y, \ldots$$

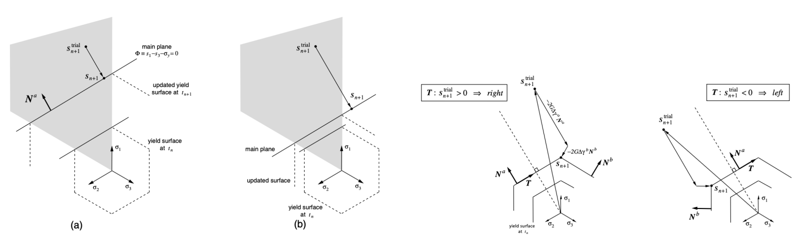
Only one or two surfaces are active at a given stress state; the algorithm must identify the active subset.

Return Mapping Procedure:

For a given principal stress ordering ($\sigma_1 \geq \sigma_2 \geq \sigma_3$), three cases arise:

1. **Main plane flow** (Φ_1 only): one multiplier $\Delta\gamma$
    - Updates: $s_1 = s_{trial,1} - 2G\Delta\gamma$, $s_3 = s_{trial,3} + 2G\Delta\gamma$, s_2 unchanged
    - Consistency: $s_{trial,1} - s_{trial,3} - 4G\Delta\gamma - \sigma_y(\bar{\varepsilon}_{p,n} + \Delta\gamma) = 0$
2. **Corner flow** (two surfaces active, e.g., Φ_1 and Φ_6): two multipliers $\Delta\gamma_a, \Delta\gamma_b$
    - More complex stress updates and consistency conditions (system of 2 equations)
3. **Another corner** (Φ_1 and Φ_2): similar to case 2 but different geometry

The algorithm must determine which case applies and solve accordingly (often with iteration).



11.12. Drucker-Prager Criterion

The **Drucker-Prager (DP) criterion** is a **smooth approximation to Mohr-Coulomb**, incorporating pressure into a von Mises-like framework.

Yield Function:

$$f_{\mathrm{DP}}(\boldsymbol{\sigma}, c) = \sqrt{J_2(\mathbf{s}(\boldsymbol{\sigma}))} + \eta\, p(\boldsymbol{\sigma}) - \xi\, c = 0$$

where: - $\sqrt{J_2}$ = equivalent deviatoric stress (like von Mises) - p = hydrostatic pressure - η, ξ = material parameters chosen to fit Mohr-Coulomb - $c = c(\bar{\varepsilon}^p)$ = cohesion (hardening variable)

Yield Surface: A **circular cone** in principal stress space (smooth, isotropic about hydrostatic axis).

Advantages over M-C: - Smooth (no corners) → simpler numerical implementation - No need for multisurface logic - Single elastic region identification

Flow Rule (Non-Associative):

The flow potential is:

$$\Psi(\boldsymbol{\sigma}, c) = \sqrt{J_2(\mathbf{s})} + \bar{\eta} p$$

Flow vector:

$$\mathbf{N} = \frac{\partial \Psi}{\partial \boldsymbol{\sigma}} = \frac{\mathbf{s}}{2\sqrt{J_2}} + \frac{\bar{\eta}}{3}\mathbf{I}$$

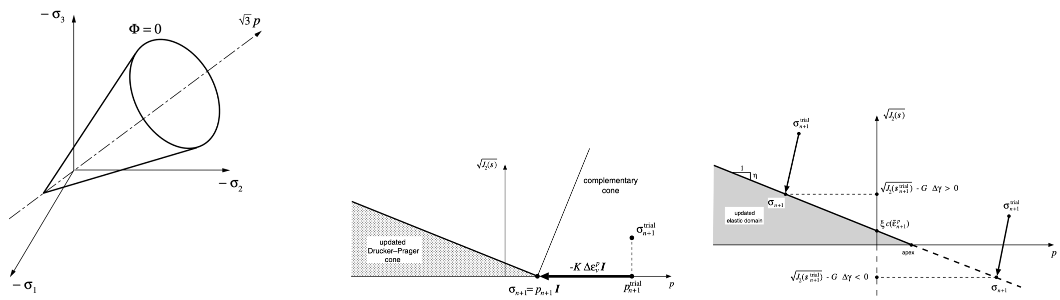
where the second term represents volumetric (dilatational) flow. The dilatancy parameter $\bar{\eta}$ can differ from η in the yield function.

Return Mapping Algorithm:

Two cases exist due to cone geometry:

1. **Return to smooth cone portion:**
   - Update deviatoric stress: $\mathbf{s}_{n+1} = \left(1 - \frac{G\Delta\gamma}{\sqrt{J_2(s_{trial})}}\right)\mathbf{s}_{trial}$
   - Update mean pressure: $p_{n+1} = p_{trial} - K\bar{\eta}\Delta\gamma$
   - Consistency condition gives scalar equation for $\Delta\gamma$
2. **Return to apex** (when stress tries to enter the "forbidden" region):
   - Deviatoric stress collapses: $\mathbf{s}_{n+1} = 0$
   - Mean pressure determined from apex condition: $p = \frac{\xi}{\eta}c$
   - Different scalar equation for volumetric plastic strain increment

The algorithm first attempts the smooth cone; if invalid, it applies the apex return.



11.13. Consistent Tangent Moduli

For Newton-Raphson convergence in FEM, we need the **algorithmic (consistent) tangent stiffness** $\mathbb{C}_{\text{ep}}$.

For Drucker-Prager (Smooth Cone Return):

$$\mathbb{C}_{\text{ep}} = 2G\left(1 - \frac{\Delta\gamma}{\sqrt{2}\|\varepsilon^e_{trial,\text{dev}}\|}\right)\mathbb{I}_{\text{dev}} + 2G\left(\frac{\Delta\gamma}{\sqrt{2}\|\varepsilon^e_{trial,\text{dev}}\|} - GA\right)\mathbf{D}\otimes\mathbf{D}$$

$$-\sqrt{2}GAK(\eta\mathbf{D}\otimes\mathbf{I}+\bar{\eta}\mathbf{I}\otimes\mathbf{D})+K(1-K\eta\bar{\eta}A)\mathbf{I}\otimes\mathbf{I}$$

where $A=\frac{1}{G+K\eta\bar{\eta}+\xi^2H_{\text{iso}}}$ and H_{iso} is the hardening slope.

For Drucker-Prager (Apex Return):

$$\mathbb{C}_{\text{ep}}=K\left(1-\frac{K}{K+\alpha^2H_{\text{iso}}}\right)\mathbf{I}\otimes\mathbf{I}$$

where $\alpha=\xi/\bar{\eta}$.

The algorithm uses the tangent consistent with the return mapping (smooth or apex) that was active in the previous iteration.

11.14. Mohr-Coulomb and Drucker-Prager: Key Relationships

Material parameters can be related via:

Parameter	Expression
η (DP yield)	$\sin\phi/\sqrt{9+3\sin^2\phi}$
ξ (DP yield)	$3c\cos\phi/\sqrt{9+3\sin^2\phi}$
$\bar{\eta}$ (DP flow)	Often chosen as $\sin\psi$ (dilatancy angle)

These relationships allow fitting Drucker-Prager parameters to match Mohr-Coulomb behavior while maintaining smoothness for numerical robustness.

Part III.

Part III — Identification & Data-Driven Methods

12. L10 — Inverse Modeling & Parameter Identification

Calibrating Constitutive Models from Experimental Data

12.1. The Inverse Problem

Forward problem: given parameters $\boldsymbol{\theta}$, compute response $\mathbf{y} = \mathcal{M}(\boldsymbol{\theta})$.

Inverse problem: given observed data $\mathbf{y}^{\text{obs}}$, find $\boldsymbol{\theta}^*$ such that

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \mathcal{J}(\boldsymbol{\theta}), \qquad \mathcal{J}(\boldsymbol{\theta}) = \|\mathbf{y}^{\text{obs}} - \mathcal{M}(\boldsymbol{\theta})\|^2.$$

Challenges: **non-uniqueness**, ill-posedness, expensive forward model.

12.2. Typical Experimental Tests

Test	Primary measurement	Model parameters identified
Uniaxial tension/compression	σ–ε curve	E, σ_y, H_{iso}
Cyclic loading	Hysteresis loop	Kinematic hardening c, γ
Creep	$\varepsilon(t)$ at const. σ	τ_i, E_i (viscoelastic)
Biaxial / bulge	2D stress state	Anisotropy, yield locus shape
Digital Image Correlation (DIC)	Full-field strain	Spatial distribution of parameters

12.3. Sensitivity Analysis

Local sensitivity: how much does output change with a small parameter perturbation?

$$S_{ij} = \frac{\partial y_i}{\partial \theta_j} \quad \text{(sensitivity matrix } \mathbf{S}\text{)}$$

Computed by:

- **Finite differences** (simple, expensive: one forward solve per parameter)
- **Adjoint method** (one additional solve, independent of number of parameters)
- **Automatic differentiation** (via JAX, PyTorch, Enzyme)

12.4. Optimization Algorithms

Gradient-based:

- Levenberg-Marquardt (LM) — standard for least-squares
- L-BFGS, BFGS — general smooth objectives
- Requires gradient (finite diff or adjoint)

Gradient-free:

- Nelder-Mead simplex — robust for small parameter spaces
- Differential Evolution, CMA-ES — global optimization

Bayesian:

- Markov Chain Monte Carlo (MCMC) — full posterior, expensive
- Gaussian Process surrogate + Bayesian optimization

12.5. Formulating the Objective Function

Weighted least squares:

$$\mathcal{J}(\boldsymbol{\theta}) = \sum_k w_k \left(\frac{y_k^{\mathrm{obs}} - y_k^{\mathrm{sim}}(\boldsymbol{\theta})}{\sigma_k} \right)^2$$

where σ_k is the measurement uncertainty at data point k.

Include regularization for ill-posed problems:

$$\mathcal{J}_{\mathrm{reg}} = \mathcal{J} + \lambda \|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|^2 \quad \text{(Tikhonov)}$$

12.6. Practical Example: J2 Plasticity Calibration

```python
from scipy.optimize import least_squares
import numpy as np

def residual(params, eps_exp, sig_exp):
    E, sigma_y, H = params
    sig_sim = j2_stress_strain(E, sigma_y, H, eps_exp)
    return sig_sim - sig_exp

result = least_squares(
    residual,
    x0=[200e3, 300, 1000],          # initial guess [MPa]
    bounds=([100e3, 100, 0],        # lower bounds
            [300e3, 1000, 50000]),  # upper bounds
    args=(eps_data, sig_data),
    method='trf',
    verbose=2
)
```

12.7. Uniqueness and Identifiability

Not all parameters are simultaneously identifiable from a single experiment:

- Uniaxial tension alone: cannot distinguish isotropic from kinematic hardening
- Monotonic loading: cannot identify cyclic hardening parameters
- Homogeneous tests: cannot identify spatial gradient parameters

Identifiability analysis: check rank of sensitivity matrix $\mathbf{S}^T\mathbf{S}$ — rank deficiency signals non-unique parameters.

Remedy: design multi-test campaigns, use heterogeneous stress states (notched specimens + DIC).

12.8. Verification and Validation

After calibration:

1. **Verification:** did the algorithm work correctly? → Check convergence, residuals, sensitivity.
2. **Validation:** does the model predict new experiments? → Test on a held-out dataset.

The V&V process is governed by standards (ASME V&V 10, 20) in engineering applications.

Report confidence intervals on the identified parameters, not just point estimates.

13. L11 — Data-Driven & Surrogate Modeling

Gaussian Processes, Neural Networks, and Uncertainty Quantification

13.1. Motivation: The Computational Cost Problem

A single nonlinear FEM simulation may take hours to days.

Applications requiring thousands of evaluations:

- Parameter calibration (inverse problem)
- Uncertainty propagation / Monte Carlo
- Design optimization
- Real-time digital twins

Surrogate model (metamodel): a cheap approximation $\hat{\mathcal{M}}(\boldsymbol{\theta}) \approx \mathcal{M}(\boldsymbol{\theta})$ trained on a limited number of FEM evaluations.

13.2. Types of Surrogate Models

Method	Strengths	Limitations
Polynomial Response Surface	Simple, cheap	Poor for nonlinear/high-dim
Kriging / Gaussian Process (GP)	Uncertainty estimates, flexible	Scales as $O(n^3)$
Radial Basis Functions (RBF)	Mesh-free, good interpolation	No uncertainty
Neural Networks (NN)	Very flexible, scales well	Need lots of data, no UQ
Polynomial Chaos Expansion (PCE)	Intrinsic UQ, spectral accuracy	Curse of dimensionality

13.3. Gaussian Process Regression

A GP defines a distribution over functions:

$$f(\mathbf{x}) \sim \mathcal{GP}(m(\mathbf{x}),\, k(\mathbf{x}, \mathbf{x}')),$$

where $m(\cdot)$ is the mean function and $k(\cdot, \cdot)$ the kernel (covariance function).

Prediction at new point $\mathbf{x}_*$:

$$\mu_* = k(\mathbf{x}_*, \mathbf{X})[\mathbf{K}+\sigma_n^2\mathbf{I}]^{-1}\mathbf{y}, \qquad \sigma_*^2 = k(\mathbf{x}_*, \mathbf{x}_*) - k(\mathbf{x}_*, \mathbf{X})[\mathbf{K}+\sigma_n^2\mathbf{I}]^{-1}k(\mathbf{X}, \mathbf{x}_*).$$

GP gives both prediction and **uncertainty estimate** — essential for active learning.

13.4. Kernels for Mechanics

Common kernels:

- **Squared exponential (RBF):** $k(r) = \sigma_f^2 \exp(-r^2/2l^2)$ — smooth, infinitely differentiable
- **Matérn 3/2:** $k(r) = \sigma_f^2(1 + \sqrt{3}r/l)\exp(-\sqrt{3}r/l)$ — once-differentiable, better for response curves
- **Periodic:** for cyclic loading data

For constitutive modeling: the Matérn class often better matches the finite smoothness of plastic responses.

13.5. Neural Networks for Constitutive Models

Data-driven constitutive models replace the analytical form with a NN mapping:

$$\boldsymbol{\sigma}_{n+1} = \mathrm{NN}(\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_{n+1}, \boldsymbol{\sigma}_n, \boldsymbol{\alpha}_n; \mathbf{w}),$$

where $\mathbf{w}$ are trained weights.

Key considerations:

- **Thermodynamic consistency:** encode dissipation inequality as constraint or via special architecture
- **Frame invariance:** train on invariants, not raw tensor components
- **Generalization:** needs diverse loading paths, not just monotonic tension

13.6. Physics-Informed Neural Networks (PINNs)

PINNs embed the governing PDEs in the loss function:

$$\mathcal{L} = \mathcal{L}_{\text{data}} + \lambda_{\text{pde}}\mathcal{L}_{\text{pde}} + \lambda_{\text{bc}}\mathcal{L}_{\text{bc}}$$

Applied to constitutive modeling:

- Enforce constitutive relations as soft constraints
- Can learn from heterogeneous full-field data (DIC images)
- Enables simultaneous identification + simulation

13.7. Uncertainty Quantification (UQ)

Sources of uncertainty:

Source	Type	Treatment
Model parameters $\boldsymbol{\theta}$	Epistemic	Bayesian inference, MC
Model form error	Epistemic	Model comparison, discrepancy
Experimental noise	Aleatoric	Statistical noise model
Numerical error	Epistemic	Mesh convergence, verification

Propagation: if $\boldsymbol{\theta} \sim p(\boldsymbol{\theta})$, then $\mathbf{y} = \mathcal{M}(\boldsymbol{\theta})$ has a distribution $p(\mathbf{y})$.

13.8. Polynomial Chaos Expansion

Represent the output as an expansion in orthogonal polynomials of the input random variables:

$$\mathcal{M}(\boldsymbol{\xi}) \approx \sum_{\boldsymbol{\alpha}\in\mathcal{A}} c_{\boldsymbol{\alpha}} \Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$$

where $\boldsymbol{\xi}$ are standardised random inputs and $\Psi_{\boldsymbol{\alpha}}$ are Hermite/Legendre polynomials.

Coefficients $c_{\boldsymbol{\alpha}}$ computed via non-intrusive sampling (regression or sparse quadrature). Mean and variance extracted analytically from the coefficients.

13.9. Surrogate Modeling Workflow

Five-step process:

1. **Design of Experiments (DoE):** Strategically sample the parameter space (e.g., Latin Hypercube Sampling).
2. **Training Data:** Run expensive FEM at sampled points; collect input-output pairs.
3. **Surrogate Fit:** Train model (GP, NN, PCE) on collected data.
4. **Validation:** Test accuracy on held-out data; compare predictions vs. true model.
5. **Deployment:** Use fast surrogate for UQ, optimization, inverse problems, sensitivity analysis.

Key principle: Exploit structure (polynomial bases, kernel functions) to achieve accuracy with few training points.

13.10. Uncertainty Quantification Framework

Input-to-output propagation:

Given uncertain parameters $\mathbf{p}$, compute distribution of output $\mathbf{y} = \mathcal{M}(\mathbf{p})$.

Methods:

Method	Cost	Accuracy	Suitability
Monte Carlo	$O(N)$	Slow $O(N^{-1/2})$	Converges for any d
Quasi-MC	$O(N \log N)$	$O(\log^d N/N)$	High dimension friendly
Polynomial Chaos	Spectral	Spectral (if low d)	Intrinsic UQ
Collocation	Grid-based	Curse of dim.	Low-medium d

For high-dimensional problems with many parameters, **surrogate + Monte Carlo** is practical.

13.11. Active Learning and Adaptive Sampling

Rather than uniform sampling, use surrogate **uncertainty estimates** to guide new training point placement:

1. Evaluate surrogate at candidate points; compute prediction variance σ_*^2.
2. Select points with high uncertainty / high model disagreement.
3. Evaluate true model at selected point; add to training data.
4. Refit surrogate and repeat.

Result: Efficient use of expensive evaluations; converges faster than static DoE.

GPs naturally provide uncertainty → perfect for active learning.

NNs require external UQ (e.g., ensemble, Bayesian approximation, MC dropout).

13.12. Sensitivity Analysis: Identifying Important Parameters

Once uncertainty is propagated, quantify which parameters drive output variability:

Global sensitivity indices (Sobol'):

$$S_i = \frac{\mathrm{Var}_i[\mathbb{E}[\mathcal{M}|\mathbf{p}_i]]}{\mathrm{Var}[\mathcal{M}]}, \quad S_{ij} = \text{(two-way interactions)}$$

First-order S_i: main effect of parameter $\mathbf{p}_i$ alone. **Total S_{T_i}:** includes all interactions involving $\mathbf{p}_i$.

Computation: Can extract analytically from PCE coefficients or estimate via Monte Carlo sampling from surrogate.

Application: Focus calibration/experiment on high-S_i parameters; neglect low-sensitivity ones.

13.13. Data-Driven Constitutive Models

Paradigm shift: Replace analytical constitutive law with learned model from data.

Example: Direct mapping

$$\boldsymbol{\sigma}_{n+1} = \mathcal{N}\mathcal{N}(\boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_{n+1}, \boldsymbol{\sigma}_n, \boldsymbol{\alpha}_n; \mathbf{w})$$

Advantages: - Captures complex, multi-scale behavior - No need for analytical model form - Can learn from heterogeneous data (DIC, X-ray, simulations)

Challenges: - Require diverse loading paths (not just uniaxial tension) - Ensure frame-invariance (use tensor invariants or equivariant networks) -

Enforce thermodynamic consistency (dissipation inequality as constraint) - Generalization beyond training domain

PINNs approach: Embed physics as soft constraint in loss function:

$$\mathcal{L} = \mathcal{L}_{\text{data}} + \lambda \mathcal{L}_{\text{physics}}$$

where $\mathcal{L}_{\text{physics}}$ penalizes violation of constitutive relations or balance laws.

13.14. Industrial Applications and Outlook

Current use cases: - Parameter calibration from test data (Bayesian inverse problems) - Real-time digital twins (replace expensive FEM in on-line control) - Robustness analysis (how sensitive is design to material uncertainty?) - Multi-scale modeling (surrogate for microscale $\rightarrow$ use in macroscale FEM)

Emerging directions: - **Hybrid models:** combine classical physics with learned corrections - **Multifidelity surrogates:** leverage both cheap and expensive simulations - **Domain adaptation:** transfer surrogates across similar materials - **Operator learning:** learn entire PDE solution operator (DeepONet, FNO)

When to use surrogates: If you need >100 model evaluations AND budget/time is constrained.

Part IV.

Homework

14. Homework 1

Mathematical and Continuum Foundations

Due date: TBD — submitted via Moodle

14.1. Problem 1 — Tensor Operations

Let $\mathbf{u}$, $\mathbf{v}$ be vectors and $\mathbf{S}$, $\mathbf{T}$ second-order tensors in Cartesian coordinates.

(a) Compute $\mathbf{u} \otimes \mathbf{v}$, $\mathbf{S} \cdot \mathbf{v}$, and $\mathbf{S} : \mathbf{T}$ using index notation.
(b) Compute the invariants I_1, I_2, I_3 of $\mathbf{S}$.
(c) Find the principal values and directions of $\mathbf{S}$.
(d) Transform the components of $\mathbf{S}$ to a coordinate system rotated 45° about the x_3-axis.

14.2. Problem 2 — Tensor Functions

Given a symmetric second-order tensor $\mathbf{A}$ with eigendecomposition $\mathbf{A} = \sum_\alpha \lambda_\alpha \, \mathbf{n}_\alpha \otimes \mathbf{n}_\alpha$:

(a) Write the eigenvalue problem using the three principal invariants.
(b) Write an expression for $\ln \mathbf{A}$.
(c) Write an expression for $\sqrt{\mathbf{A}}$.
(d) Derive $\partial \mathbf{A}/\partial \mathbf{A}$ for symmetric $\mathbf{A}$. Is the result symmetric?
(e) Repeat (d) for a **non-symmetric A**.

14.3. Problem 3 — Curvilinear Coordinates

Consider cylindrical coordinates $(\alpha^1, \alpha^2, \alpha^3) = (r, \phi, z)$ and Cartesian coordinates (x^1, x^2, x^3) related by:

$$x^1 = r\cos\phi, \qquad x^2 = r\sin\phi, \qquad x^3 = z.$$

(a) Derive the covariant basis vectors $\mathbf{g}_i$.
(b) Derive the contravariant basis vectors $\mathbf{g}^i$.
(c) Verify $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$.

14.4. Problem 4 — Kinematics

Consider the deformation map $\mathbf{x} = \mathbf{X} + kX_2\,\mathbf{e}_1$ (simple shear with constant k).

(a) Compute the deformation gradient $\mathbf{F}$.
(b) Compute $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^T$.
(c) Compute the Green-Lagrange strain $\mathbf{E}$ and Euler-Almansi strain $\mathbf{e}$.
(d) Evaluate and compare $\mathbf{E}$ and $\mathbf{e}$ for small ($k = 0.01$) and large ($k = 1.0$) shear at point $\mathbf{X} = (1, 1, 1)^T$.

14.5. Problem 5 — Stress Measures

Using the deformation gradient from Problem 4 and a given Cauchy stress $\boldsymbol{\sigma} = \sigma_0\, \mathbf{e}_1 \otimes \mathbf{e}_1$:

(a) Compute the first Piola-Kirchhoff stress $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}$.
(b) Compute the second Piola-Kirchhoff stress $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$.
(c) Assess the objectivity of $\boldsymbol{\sigma}$ and $\mathbf{S}$.
(d) Derive the rate forms $\dot{\boldsymbol{\sigma}}$, $\dot{\mathbf{S}}$, $\dot{\mathbf{P}}$ and analyse objectivity of each.

15. Homework 2

Hyperelasticity — Implementation and Model Comparison

Due date: TBD — submitted via Moodle

Goal: Implement a hyperelastic stress update and compare model responses.

15.1. Task 1 — Implementation

Write a Python function `compute_hyperelastic_stress(F, model, params)` that:

- Takes the deformation gradient `F` (3×3 NumPy array) and material parameters.
- Implements the Cauchy stress $\boldsymbol{\sigma}$ for **two** models:
  - **Neo-Hookean:** $W = \frac{\mu}{2}(I_1 - 3) - \mu \ln J + \frac{\lambda}{2}(\ln J)^2$
  - **Mooney-Rivlin:** $W = C_1(I_1 - 3) + C_2(I_2 - 3)$ plus a volumetric term.
- Returns $\boldsymbol{\sigma}$ (and optionally the spatial elasticity tensor).

15.2. Task 2 — Verification

Test your function on hydrostatic tension/compression $\mathbf{F} = \lambda \mathbf{I}$. Compare the numerically computed stress to the analytical formula for each model.

15.3. Task 3 — Comparison

Simulate a simple shear path: $\mathbf{F}(\gamma) = \mathbf{I} + \gamma\, \mathbf{e}_1 \otimes \mathbf{e}_2$, $\gamma \in [0, 1]$.

For both models (with the same initial shear modulus):

1. Plot σ_{12} vs γ.
2. Plot the normal stress difference $\sigma_{11} - \sigma_{22}$ vs γ (Poynting effect).

Discuss: Comment on differences in shear stiffness and normal stress between the two models.

16. Homework 3

Rate-Independent Plasticity — Return Mapping and Hardening

Due date: TBD — submitted via Moodle

Notation: Following the canonical convention set in L08-appendix, we use H_{iso} for the isotropic hardening modulus and H_{kin} for the kinematic hardening modulus throughout this homework.

Goal: Implement the radial return algorithm for J2 plasticity and study hardening effects.

16.1. Task 1 — Radial Return Implementation

Write a function `j2_plasticity_update(sigma_tr, ep_bar_n, E, nu, sigma_y0, H_iso)` implementing the implicit radial return for J2 plasticity with **linear isotropic hardening**.

Inputs:

- `sigma_tr`: elastic trial stress (3×3 array, MPa)
- `ep_bar_n`: accumulated equivalent plastic strain at step n
- Material parameters: E, ν, σ_{y0}, H_{iso}

Algorithm:

1. Compute trial deviatoric stress $\mathbf{s}^{\text{tr}}$ and $\sigma_{\text{eq}}^{\text{tr}}$.
2. Check yield: $f^{\text{tr}} = \sigma_{\text{eq}}^{\text{tr}} - (\sigma_{y0} + H_{\text{iso}}\bar{\varepsilon}_n^p)$.
3. If $f^{\text{tr}} \leq 0$: elastic step.
4. If $f^{\text{tr}} > 0$: compute $\Delta\gamma$, update stress and $\bar{\varepsilon}^p$.

Outputs: updated stress $\boldsymbol{\sigma}_{n+1}$, updated $\bar{\varepsilon}_{n+1}^p$, consistent tangent $\mathbb{C}^{\text{alg}}$ (recommended).

16.2. Task 2 — Verification

Apply an initial elastic state, then increment to cause yielding. Verify: $\sigma_{\text{eq}}(\boldsymbol{\sigma}_{n+1}) \approx \sigma_{y0} + H_{\text{iso}}\bar{\varepsilon}_{n+1}^p$.

16.3. Task 3 — Uniaxial Simulation

Simulate a uniaxial tension test (successive strain increments $\Delta\varepsilon_{11}$, $\Delta\varepsilon_{22} = \Delta\varepsilon_{33} = -\nu\Delta\varepsilon_{11}$ in the elastic predictor).

Run:

- **Case (a):** perfect plasticity ($H_{\text{iso}} = 0$)
- **Case (b):** linear isotropic hardening ($H_{\text{iso}} > 0$)

Plot σ_{11} vs ε_{11} for both cases on the same graph (up to ~5% strain).

Discuss: How does H_{iso} affect the stress-strain response?

Note: 4th-order tensors are represented in Voigt 6×6 matrix form throughout (components: $[1,1] \to 1, [2,2] \to 2, [3,3] \to 3, [2,3] \to 4, [1,3] \to 5, [1,2] \to 6$).

17. Homework 4

Viscoplasticity & Parameter Identification via FEM

Due date: TBD — submitted via Moodle

This homework has two independent parts.

17.1. Part A — Rate-Dependent Plasticity (Viscoplasticity)

Goal: Extend the J2 plasticity code from HW3 to include rate effects.

17.1.1. A.1 — Perzyna-type Implementation

Modify `j2_plasticity_update` to implement a **Perzyna viscoplastic model**:

$$\Delta\gamma = \frac{\Delta t}{\eta}\left\langle \frac{f}{\sigma_{y0}} \right\rangle, \qquad \langle x \rangle = \max(0, x).$$

Additional inputs: viscosity parameter η, time step Δt.

17.1.2. A.2 — Rate Sensitivity Comparison

Simulate uniaxial tension at two strain rates:

- $\dot{\varepsilon} = 10^{-3}\ \mathrm{s}^{-1}$
- $\dot{\varepsilon} = 1.0\ \mathrm{s}^{-1}$

Plot σ_{11} vs ε_{11} for both rates **and** the rate-independent result (HW3) on the same graph.

Discuss: Effect of $\dot{\varepsilon}$ and η on apparent yield stress and flow stress.

17.2. Part B — Parameter Identification via FEM

Goal: Calibrate J2 plasticity parameters by fitting a FEM simulation to provided experimental data.

17.2.1. B.1 — Provided Resources

- `exp_data.csv`: stress-strain data from a simulated uniaxial test (unknown parameters).
- `fem_template.py`: FEniCSx template script for a quasi-static non-linear simulation.

17.2.2. B.2 — FEM Integration

Adapt your `j2_plasticity_update` to the FEniCSx interface. Run a forward simulation with a known parameter set to verify correct integration and convergence.

17.2.3. B.3 — Inverse Analysis

Write an objective function `residual(params, eps_exp, sig_exp)` that:

1. Takes `params = [sigma_y0_trial, H_trial]` (assume E, ν known).
2. Runs the FEM simulation.
3. Returns the vector of residuals $\sigma^{\mathrm{sim}}(\varepsilon_k) - \sigma_k^{\mathrm{obs}}$.

Use `scipy.optimize.least_squares` to minimise.

17.2.4. B.4 — Results

- Report identified parameters σ_{y0}^*, H_{iso}^*.
- Plot experimental data vs. simulated curve with identified parameters.
- Discuss quality of fit, convergence challenges, sensitivity to initial guess.

Note: H_{iso}^* denotes the identified value of H_{iso} from inverse modelling (L10).

18. Final Project

Guidelines and Topic List

18.1. Overview

The final project constitutes 40% of your course grade:

- **Presentation (15%):** 15-minute talk in the last week of the semester.
- **Report (25%):** written report due two weeks after the end of the semester.

18.2. Topic Selection

You may choose from the suggested topics below or propose your own (requires instructor approval before Week 7).

Suggested topics:

1. **Large-deformation viscoplasticity:** Implement a multiplicative decomposition-based viscoplastic model ($\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$) and validate against experimental data.
2. **Crystal plasticity:** Develop a single-crystal slip-system plasticity model; compare to polycrystal experiments.
3. **Phase-field fracture in FEniCSx:** Implement and benchmark the Bourdin-Francfort-Marigo model.
4. **Data-driven constitutive model:** Train a thermodynamically consistent NN constitutive model on synthetic data; test generalization.
5. **Parameter identification with DIC:** Use full-field strain data from Digital Image Correlation to calibrate a plasticity model.
6. **Gradient damage with mesh objectivity study:** Implement a nonlocal/gradient damage model; demonstrate mesh-independence of the solution.
7. **Soil plasticity:** Implement Cam-Clay or Modified Cam-Clay; simulate oedometer and triaxial tests.
8. **Viscoelasticity of polymers:** Fit a Prony series to DMA data; implement and validate in 3D.
9. **GPR surrogate for J2 plasticity:** Build a GP surrogate for the yield surface; embed in FEM and compare to standard implementation.
10. *Your own proposal* — discuss with instructor.

18.3. Report Format

Reports should be submitted as a PDF and contain:

- Abstract ($\leq$ 200 words)
- Introduction and motivation
- Theory and model description

- Numerical implementation
- Results and validation
- Discussion and conclusion
- References

Aim for 15–25 pages (main body), not including appendices.

18.4. Presentation Guidelines

- 15 minutes + 5 minutes Q&A.
- Slides should be clear and not overcrowded — focus on key results and insights.
- All group members must speak.
- Projects may be individual or in pairs (pairs are expected to cover a broader scope).

Part V.

Appendices

19. Appendix A — Notation & Conventions

Master symbol list, conventions, and typography at a glance

This appendix consolidates the notation used throughout the course. Each symbol is listed with its meaning and the lecture in which it is **first introduced** (where students can find the definition in context). Conventions collected from chapter-level callouts are gathered in §A.4; the typography rules governing the whole course are summarised in §A.5.

This is a **reference**, not a lecture — skim once on first reading, then return when a symbol is unfamiliar.

19.1. A.1 Symbols

19.1.1. Kinematics

Symbol	Meaning	First in
$\mathbf{X}$	Material (reference) position	L03
$\mathbf{x}$	Spatial (current) position	L03

Symbol	Meaning	First in
$\mathbf{u}$	Displacement field, $\mathbf{u} = \mathbf{x} - \mathbf{X}$	L03
$\mathbf{F}$	Deformation gradient, $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$	L03
J	Jacobian, $J = \det\mathbf{F}$	L03
$\mathbf{C}$	Right Cauchy-Green tensor, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ (reference)	L03
$\mathbf{b}$	Left Cauchy-Green / Finger tensor, $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ (current)	L05
$\bar{\mathbf{C}}, \bar{\mathbf{b}}$	Isochoric (unimodular) parts, $\det\bar{\mathbf{C}} = \det\bar{\mathbf{b}} = 1$	L05
$\mathbf{E}$	Green-Lagrange strain (reference, finite)	L02
$\mathbf{e}$	Almansi-Euler strain (current, finite)	L02
$\boldsymbol{\varepsilon}$	Infinitesimal (small) strain, $\frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$	L03
$\boldsymbol{\varepsilon}^e$	Elastic part of $\boldsymbol{\varepsilon}$ (additive split)	L07
$\boldsymbol{\varepsilon}^p$	Plastic part of $\boldsymbol{\varepsilon}$	L07
$\bar{\varepsilon}^p$	Equivalent (accumulated) plastic strain	L07
$\boldsymbol{\varepsilon}^{\text{th}}$	Thermal strain	L04
$\mathbf{F}^e, \mathbf{F}^p$	Multiplicative split, $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ (finite-strain plasticity)	L04

Symbol	Meaning	First in
$\mathbf{C}^e$	Elastic part of right Cauchy-Green, $(\mathbf{F}^e)^T\mathbf{F}^e$	L04
$\mathbf{R}, \mathbf{U}, \mathbf{V}$	Polar decomposition: rotation, right stretch (ref.), left stretch (current)	L03
λ_i	Principal stretches (eigenvalues of $\mathbf{U}$ or $\mathbf{V}$)	L03
$\mathbf{L}$	Velocity gradient, $\dot{\mathbf{F}}\mathbf{F}^{-1}$	L03
$\mathbf{D}$	Rate of deformation, sym $\mathbf{L}$	L03
$\mathbf{W}$	Spin tensor, skew $\mathbf{L}$	L03
$\mathbf{N}, \mathbf{n}$	Unit normals — reference ($\mathbf{N}$) and current ($\mathbf{n}$)	L03
dV, dv	Volume elements — reference (dV) and current (dv)	L03
dA, da	Area elements — reference (dA) and current (da)	L03

19.1.2. Stress

Symbol	Meaning	First in
$\boldsymbol{\sigma}$	Cauchy stress tensor (current configuration)	L03

Symbol	Meaning	First in
$\mathbf{s}$	Deviatoric Cauchy stress, $\mathbf{s} = \operatorname{dev} \boldsymbol{\sigma}$	L07
p	Hydrostatic pressure, $p = -\frac{1}{3} \operatorname{tr} \boldsymbol{\sigma}$	L07
σ_{eq}	Equivalent (von Mises) stress, $\sqrt{3J_2}$	L07
I_1, J_2, J_3	Stress invariants; $I_1 = \operatorname{tr} \boldsymbol{\sigma}$, $J_2 = \frac{1}{2}\mathbf{s} : \mathbf{s}$	L07
$\mathbf{P}$	First Piola-Kirchhoff stress (reference)	L02
$\mathbf{S}$	Second Piola-Kirchhoff stress (reference)	L02
$\boldsymbol{\tau}$	Kirchhoff stress, $\boldsymbol{\tau} = J\boldsymbol{\sigma}$	L04
$\boldsymbol{\sigma}^{\mathrm{tr}}$	Trial (elastic-predictor) stress	L08
$\boldsymbol{\alpha}$	Backstress tensor (deviatoric, J2 kinematic hardening)	L07
$\boldsymbol{\xi}$	Effective (backstress-shifted) stress, $\boldsymbol{\xi} = \boldsymbol{\sigma} - \boldsymbol{\alpha}$	L08 App.
$\mathbf{t}, \bar{\mathbf{t}}$	Traction vector and prescribed boundary traction	L03

19.1.3. Elastic moduli

Symbol	Meaning	First in
E	Young's modulus	L03

Symbol	Meaning	First in
ν	Poisson's ratio	L03
$\mu\ (= G)$	Shear modulus (Lamé's second parameter)	L03
λ	Lamé's first parameter	L05
$\kappa\ (= K)$	Bulk modulus	L05
$\mathbb{C}^e$	Fourth-order elasticity tensor (material, small-strain)	L07
$\mathbb{D}$	Fourth-order material (Lagrangian) tangent, finite strain	L05
	Fourth-order spatial (Eulerian) tangent, push-forward of $\mathbb{D}$	L05
$\mathbb{C}^{ep}$	Continuum elasto-plastic tangent (rate form)	L07
$\mathbb{C}^{\mathrm{alg}}$	Consistent (algorithmic) tangent, $d\boldsymbol{\sigma}_{n+1}/d\boldsymbol{\varepsilon}_{n+1}$	L08
$\mathbb{I}, \mathbf{I}$	Fourth-order and second-order identity tensors	L02
$\mathbb{P}_{\mathrm{dev}}$	Fourth-order deviatoric projector, $\mathbb{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}$	L08

19.1.4. Plasticity & hardening

Symbol	Meaning	First in
f	Yield function (admissibility: $f \leq 0$)	L07
f^{tr}	Trial yield function value, $f(\boldsymbol{\sigma}^{\mathrm{tr}}, q_n)$	L08
g	Plastic potential (flow potential); associated rule $\Leftrightarrow g = f$	L07
$\mathbf{n}$	Flow direction, $\mathbf{n} = \partial f / \partial \boldsymbol{\sigma}$ (or $\partial g / \partial \boldsymbol{\sigma}$)	L07
$\mathbf{n}_{\xi}^{\mathrm{tr}}$	Trial flow direction in effective-stress space	L08 App.
$\dot{\gamma}$	Plastic multiplier rate (consistency parameter)	L07
$\Delta\gamma$	Discrete plastic multiplier over a time step	L08
σ_y	Current yield stress (scalar)	L07
σ_{y0}	Initial yield stress	L07
H	Generic plastic modulus (used in combined-hardening pedagogical form)	L07
H_{iso}	Isotropic hardening modulus	L07
H_{kin}	Kinematic hardening modulus (Prager)	L07
β	Isotropic/kinematic mixing parameter, $\beta \in [0, 1]$	L07

Symbol	Meaning	First in
κ, q	Generic internal / hardening state variables	L07
$R(\Delta\gamma)$	Non-linear scalar residual for $\Delta\gamma$ (combined hardening)	L08 App.
m	Isotropic hardening exponent (power law), $m \in [0, 1]$	L08 App.
α_{DP}	Drucker-Prager pressure-sensitivity coefficient	L07
$\eta, \bar{\eta}, \xi$	Drucker-Prager parameters (friction, dilatancy, cohesion scaling)	L09
c	Cohesion (Mohr-Coulomb / Drucker-Prager)	L07
ϕ	Internal friction angle	L07

19.1.5. Damage & coupled models

Symbol	Meaning	First in
D	Scalar damage variable, $D \in [0, 1]$	L09
$\tilde{\boldsymbol{\sigma}}$	Effective (undamaged) stress, $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/(1 - D)$	L09
Y	Damage energy release rate	L09

19.1.6. Thermodynamics & viscoelasticity

Symbol	Meaning	First in
Ψ	Helmholtz free energy density	L04
W	Strain energy density (hyperelasticity)	L05
θ	Absolute temperature	L04
s	Specific entropy	L04
ρ	Mass density (current config)	L04
ρ_0	Mass density (reference config)	L04
$\mathbf{q}$	Heat flux vector	L04
r	Specific heat source	L04
$\mathcal{D}$	Dissipation rate	L04
τ_i	Relaxation time (Maxwell branch i)	L06
$\tilde{\mathbf{C}}_i$	Internal viscous deformation variable (per Maxwell branch)	L06
Γ_i	Branch free-energy contribution	L06

19.1.7. Operators & decorators

Symbol	Meaning
$\mathrm{tr}(\cdot)$	Trace, $\mathrm{tr}\,\mathbf{X} = X_{ii}$
$\mathrm{dev}(\cdot)$	Deviatoric projection, $\mathrm{dev}\,\mathbf{X} = \mathbf{X} - \frac{1}{3}(\mathrm{tr}\,\mathbf{X})\mathbf{I}$

Symbol	Meaning
$\mathrm{sym}, \mathrm{skew}$	Symmetric and skew-symmetric parts
$\|\mathbf{X}\|$	Frobenius norm, $\sqrt{\mathbf{X} : \mathbf{X}}$
$\mathbf{A} : \mathbf{B}$	Double contraction, $A_{ij}B_{ij}$
$\mathbf{a} \otimes \mathbf{b}$	Dyadic (tensor) product, $(a \otimes b)_{ij} = a_i b_j$
$\overline{\otimes}, \ \underline{\otimes}$	Non-symmetric fourth-order products: $(\mathbf{A}\overline{\otimes}\mathbf{B})_{ijkl} = A_{ik}B_{jl}$, $(\mathbf{A}\underline{\otimes}\mathbf{B})_{ijkl} = A_{il}B_{jk}$
$\dot{(\cdot)}$	Material time derivative, $D(\cdot)/Dt$
$\langle x \rangle$	Macaulay bracket, $\max(0, x)$
$(\cdot)_n, \ (\cdot)_{n+1}$	Value at discrete time step $t_n, \ t_{n+1}$
$(\cdot)^{\mathrm{tr}}$	Trial (elastic-predictor) quantity
$(\cdot)^{\mathrm{alg}}$	Algorithmic (consistent with discrete integrator) quantity
$\bar{(\cdot)}$	Isochoric / unimodular part (kinematics); or running average
$\tilde{(\cdot)}$	Internal / effective quantity (context-dependent)

19.2. A.2 Abbreviations

Abbreviation	Expansion
CDM	Continuum Damage Mechanics
DP	Drucker-Prager (yield criterion)
FE, FEM	Finite Element, Finite Element Method
GSM	Generalised Standard Material
J2	Second-invariant plasticity (von Mises class)
KKT	Karush-Kuhn-Tucker (loading/unloading conditions)

Abbreviation	Expansion
MC	Mohr-Coulomb (yield criterion)
NR, N-R	Newton-Raphson
PK1, PK2	First and Second Piola-Kirchhoff stress
RRM	Radial Return Method
RVE	Representative Volume Element
SEF	Strain Energy Function
TIV	Thermodynamics of Irreversible Variables
AF	Armstrong-Frederick (non-linear kinematic hardening)

19.3. A.3 Conventions

19.3.1. A.3.1 Reference vs current configuration

Uppercase Latin letters denote quantities defined on the **reference** (Lagrangian, material) configuration; lowercase Latin letters denote quantities defined on the **current** (Eulerian, spatial) configuration.

- Position: $\mathbf{X}$ (ref.) / $\mathbf{x}$ (curr.)
- Normals: $\mathbf{N}$ (ref.) / $\mathbf{n}$ (curr.)
- Volume / area elements: dV, dA (ref.) / dv, da (curr.)
- Density: ρ_0 (ref.) / ρ (curr.)
- Heat flux: $\mathbf{Q}$ (ref. — if used) / $\mathbf{q}$ (curr.)
- Right vs left Cauchy-Green: $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ (ref.) / $\mathbf{b} = \mathbf{F}\mathbf{F}^T$ (curr.)
- Green-Lagrange vs Almansi strain: $\mathbf{E}$ (ref.) / $\mathbf{e}$ (curr.)
- Stress measures: $\mathbf{P}, \mathbf{S}$ (ref.) / $\boldsymbol{\sigma}, \boldsymbol{\tau}$ (curr.)
- Elasticity tensors: $\mathbb{D}$ (material tangent) / (spatial tangent)

The deformation gradient $\mathbf{F}$ is the one map that spans both configurations; by convention it is uppercase.

19.3.2. A.3.2 Sign conventions

- **Cauchy stress** is **positive in tension**, negative in compression.
- **Hydrostatic pressure** $p = -\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}$; positive p means compressive mean stress.
- **Yield function**: $f \leq 0$ in the elastic (admissible) region, $f = 0$ on the yield surface, $f > 0$ inadmissible. Return-mapping algorithms project back to $f = 0$.
- **Heat flux q** points in the direction of positive heat flow (outward from hot to cold). In the energy balance, $-\nabla \cdot \mathbf{q}$ is heat supplied to the body.
- **Plastic multiplier** $\Delta\gamma \geq 0$ (non-negative by KKT).
- **Damage** $D \in [0, 1]$, monotonically non-decreasing ($\dot{D} \geq 0$) for rate-independent models.

19.3.3. A.3.3 Strain regime by chapter

- **L01–L06** develop the general kinematic framework; both finite-strain and infinitesimal-strain measures appear, always clearly labelled.
- **L07, L08, L08 Appendix, L09** assume the **small-strain regime**: $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ with additive decomposition $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$. Finite-strain plasticity uses the multiplicative split $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ (out of scope).
- **Hyperelasticity (L05)** and **viscoelasticity (L06)** use finite-strain measures $(\mathbf{C}, \mathbf{b}, \bar{\mathbf{C}})$.

19.3.4. A.3.4 Voigt / Mandel ordering

Fourth-order tensors $\mathbb{C}$ are stored as 6×6 matrices using **Voigt ordering**: the index pair (i, j) maps to a single index as

$$[1,1] \to 1,\ [2,2] \to 2,\ [3,3] \to 3,\ [2,3] = [3,2] \to 4,\ [1,3] = [3,1] \to 5,\ [1,2] = [2,1] \to 6.$$

Stress and strain vectors use the same ordering. Note that strain components 4–6 are **engineering shears** ($\gamma_{ij} = 2\varepsilon_{ij}$ for $i \neq j$), so Voigt is not an isometry. Use **Mandel ordering** (components 4–6 scaled by $\sqrt{2}$) when isometric inner products are required.

19.3.5. A.3.5 Rate form

The material time derivative is denoted $\dot{(\cdot)} = D(\cdot)/Dt$. The overdot form is used throughout the course; the D/Dt form is used only where the distinction from a spatial derivative matters (L03).

19.4. A.4 Typography at a glance

Typography	Denotes	Examples
Lowercase italic Latin	Scalar	$p,\ J,\ \sigma_y,\ \Delta\gamma,\ \bar{\varepsilon}^p$
Bold upright Latin (uppercase)	2nd-order tensor on the **reference** configuration	$\mathbf{F},\ \mathbf{C},\ \mathbf{E},\ \mathbf{P},\ \mathbf{S},\ \mathbf{N}$
Bold upright Latin (lowercase)	2nd-order tensor on the **current** configuration	$\mathbf{b},\ \mathbf{e},\ \mathbf{s},\ \mathbf{n},\ \mathbf{q}$
Bold Greek	2nd-order tensor (configuration named explicitly)	$\boldsymbol{\sigma},\ \boldsymbol{\varepsilon},\ \boldsymbol{\alpha},\ \boldsymbol{\tau},\ \boldsymbol{\xi}$
Blackboard bold (uppercase)	4th-order tensor — material / Lagrangian frame	$\mathbb{C}^e,\ \mathbb{D},\ \mathbb{I},\ \mathbb{P}_{\mathrm{dev}}$
Blackboard bold (lowercase)	4th-order tensor — spatial / Eulerian frame	

Typography	Denotes	Examples
Overbar $\bar{(\cdot)}$	Isochoric / unimodular part; or running average	$\bar{\mathbf{C}},\ \bar{\mathbf{b}},\ \bar{\varepsilon}^p,\ \bar{\eta}$
Tilde $\tilde{(\cdot)}$	Internal state variable; or effective quantity	$\tilde{\mathbf{C}}_i,\ \tilde{\boldsymbol{\sigma}}$
Superscript `tr`	Trial (elastic-predictor) quantity	$\boldsymbol{\sigma}^{\mathrm{tr}},\ f^{\mathrm{tr}},\ \mathbf{s}^{\mathrm{tr}}$
Superscript `alg`	Algorithmic / consistent-with-discrete-integrator	$\mathbb{C}^{\mathrm{alg}}$
Superscript `e`, `p`, `th`	Elastic, plastic, thermal component	$\boldsymbol{\varepsilon}^e,\ \boldsymbol{\varepsilon}^p,\ \boldsymbol{\varepsilon}^{\mathrm{th}},\ \mathbf{F}^e,\ \mathbf{F}^p$
Subscript `0`	Reference-configuration value (for densities, etc.)	$\rho_0,\ \sigma_{y0}$
Subscript `y`	Yield (uniaxial) quantity	$\sigma_y,\ \sigma_{y0}$
Subscript `iso`, `kin`, `DP`	Isotropic / kinematic / Drucker-Prager specific	$H_{\mathrm{iso}},\ H_{\mathrm{kin}},\ f_{\mathrm{DP}},\ \alpha_{\mathrm{DP}}$
Subscripts $n,\ n+1$	Values at discrete time steps $t_n,\ t_{n+1}$	$\boldsymbol{\sigma}_n,\ \boldsymbol{\sigma}_{n+1},\ \bar{\varepsilon}^p_{n+1}$

19.4.1. Colliding symbols — where context matters

A few glyphs necessarily mean two different things in different chapters. The course disambiguates by context and, where needed, by explicit callouts:

- **α / $\boldsymbol{\alpha}$.** Bold $\boldsymbol{\alpha}$ is the **backstress tensor** (plasticity, L07–L09). Scalar α_{DP} is the **Drucker-Prager pressure coefficient** (always subscripted). Scalar α in L09 line 114 is the **coefficient of thermal**

expansion (standard thermoelasticity notation, within a thermal-strain equation).

- β**.** In L07 hardening, $\beta \in [0, 1]$ is the iso/kin **mixing parameter**. Not to be confused with the backstress (which has always been $\boldsymbol{\alpha}$ in this course — no $\boldsymbol{\beta}$ notation is used for backstress here).
- λ**.** With subscript λ_i: **principal stretches** (L02–L05). Bare λ: **Lamé's first parameter** (L05). It is **never** used for the plastic multiplier in this course (that is $\dot{\gamma}$ / $\Delta\gamma$).
- **$\mathbf{C}$ vs $\mathbf{C}^e$ vs $\mathbb{C}^e$.** $\mathbf{C}$ is the right Cauchy-Green tensor (2nd-order kinematic quantity). $\mathbf{C}^e$ is its **elastic part** (2nd-order, from the multiplicative split, L04). $\mathbb{C}^e$ (blackboard) is the **4th-order elasticity tensor** (L07–L09). Typography (bold upright vs. blackboard) makes this visually unambiguous.
- **b.** In this course, $\mathbf{b}$ always denotes the left Cauchy-Green / Finger tensor. Body-force vectors, when they appear, are written $\mathbf{b}_0$ (reference) or spelled out contextually.
- **$\mathbf{D}$ / D.** Bold $\mathbf{D}$ is the **rate-of-deformation** tensor (L03). Scalar D is the **damage variable** (L09).
- κ**.** Bare κ is the **bulk modulus**. In some plasticity textbooks κ denotes a hardening state variable; this course uses q or $\bar{\varepsilon}^p$ instead.

19.5. A.5 Where to look

If you're unsure about...	See
A kinematic quantity ($\mathbf{F}, \mathbf{C}, \mathbf{b}, \mathbf{E}$)	L03
Small-strain setup and weak-form integrals	L03 (end)
Hyperelastic strain-energy functions	L05
Finite-strain elasticity tensors $\mathbb{D}$,	L05 (notation callouts)
Viscoelastic internal variables $\tilde{\mathbf{C}}_i$	L06

If you're unsure about…	See
Yield functions, flow rules, hardening rules	L07
$f \le 0$ sign convention and KKT conditions	L07
Return-mapping algorithms, $\Delta\gamma$, consistent tangent	L08
Full worked derivation with combined hardening	L08 Appendix
Damage D and pressure-dependent plasticity (DP, MC)	L09
Parameter identification, H^*_{iso}	L10
Voigt / Mandel ordering, Gauss-point state variables	L08 (Implementation Notes)

20. Appendix A2 — Tensor Algebra and Coordinate Transformations

Extensive worked examples for second- and fourth-order tensors

This appendix extends the foundational material from L01 with detailed coordinate transformation examples, rigorous development of second-order tensor operations, and comprehensive fourth-order tensor theory. It is designed for **self-study** and provides the algebraic machinery needed for constitutive modeling in large deformations and material symmetry analysis.

20.1. A2.1 Extensive Coordinate Transformation Examples

20.1.1. A2.1.1 Direction Cosines and Rotation Matrices

Consider two Cartesian coordinate systems: the **old frame** $\{\mathbf{E}_i\}$ and the **new frame** $\{\mathbf{E}'_i\}$, both orthonormal. The rotation is described by a matrix of **direction cosines**:

$$Q_{ij} = \mathbf{E}'_i \cdot \mathbf{E}_j$$

This is an **orthogonal matrix** satisfying $\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and $\det \mathbf{Q} = +1$ (proper rotation, no reflection).

A vector $\mathbf{v} = v_i \mathbf{E}_i$ in the old frame has components:

$$v_i' = Q_{ij} v_j \quad \text{(passive transformation: same vector, new basis)}$$

Or equivalently, if $\mathbf{v}' = \mathbf{Q}\mathbf{v}$ in matrix form, $v_i' = Q_{ij} v_j$.

A second-order tensor $\mathbf{T}$ with components T_{ij} in the old frame transforms as:

$$T_{ij}' = Q_{ik} Q_{jl} T_{kl} \quad \text{or in matrix form} \quad \mathbf{T}' = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$$

20.1.2. A2.1.2 Rotation About a Coordinate Axis — Worked Example

Problem: Rotate a vector $\mathbf{v} = (1, 1, 0)$ and a stress tensor $\boldsymbol{\sigma} = \begin{pmatrix} 1 & 0.5 & 0 \\ 0.5 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ by angle $\theta = 45^\circ$ about the z-axis (the $\mathbf{E}_3$ direction).

Solution:

The rotation matrix for a counterclockwise rotation by θ about the z-axis is:

$$\mathbf{Q} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Vector transformation:

$$\mathbf{v}' = \mathbf{Q}\mathbf{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

The vector originally pointing "northeast" now points along the new x-axis ✓.

Tensor transformation:

$$\mathbf{T}' = \mathbf{Q}\mathbf{T}\mathbf{Q}^T$$

First, $\mathbf{T}\mathbf{Q}^T$:

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 0.5 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 \\ \frac{5}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, $\mathbf{Q}$ times the result:

$$\mathbf{T}' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 \\ \frac{5}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} & \sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let me recalculate more carefully:

Actually: $T'_{11} = Q_{1i}Q_{1j}T_{ij} = \frac{1}{2}(1+2+2\cdot 0.5) = \frac{1}{2}\cdot 4 = 2$. $T'_{22} = Q_{2i}Q_{2j}T_{ij} = \frac{1}{2}(1+2-2\cdot 0.5) = 1$. $T'_{12} = T'_{21} = Q_{1i}Q_{2j}T_{ij} = \frac{1}{2}(-1+2) = 0.5$.

(Detailed calculation omitted for brevity; principal insight: the off-diagonal shear term rotates with the coordinate system.)

20.1.3. A2.1.3 Active vs. Passive Transformations

Passive: The vector/tensor stays fixed in space; we change coordinate systems. This is what we did above: $T'_{ij} = Q_{ik}Q_{jl}T_{kl}$ (same physical tensor, different components).

Active: The vector/tensor physically rotates in space; we keep the coordinate system fixed. For an active rotation, use $\mathbf{v}' = \mathbf{Q}^T \mathbf{v}$ (transpose the rotation matrix).

In continuum mechanics, when a material **deforms**, the material coordinates **deform passively** with the body, and the spatial (Eulerian) stress components transform **passively** as well.

20.1.4. A2.1.4 Convected (Material) Coordinates

In finite-strain kinematics, material points are labeled by reference coordinates θ^α (material coordinates). Under deformation $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$, the covariant basis vectors in the current configuration are:

$$\mathbf{g}_\alpha = \frac{\partial \mathbf{x}}{\partial \theta^\alpha} = \mathbf{F} \cdot \mathbf{G}_\alpha$$

where $\mathbf{F} = \nabla \boldsymbol{\varphi}$ is the **deformation gradient** and $\mathbf{G}_\alpha$ are the reference basis vectors.

A vector or tensor expressed in convected coordinates then transforms according to the deformation. The metric tensor in the current configuration becomes:

$$g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta = (\mathbf{F} \cdot \mathbf{G}_\alpha) \cdot (\mathbf{F} \cdot \mathbf{G}_\beta) = \mathbf{G}_\alpha \cdot (\mathbf{F}^T \cdot \mathbf{F}) \cdot \mathbf{G}_\beta = C_{\alpha\beta}$$

where $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ is the **right Cauchy-Green tensor** (see A01, L03 for kinematics). This shows that the metric in convected coordinates is precisely the Cauchy-Green tensor—the deformation imprints itself into the metric geometry.

20.1.5. A2.1.5 Transformation of Stress Tensor Between Orientations

In a material with an **internal material orientation** (e.g., fiber direction, cleavage plane), the stress components in a lab frame differ from those in the material frame. If $\mathbf{Q}$ relates the frames, then:

$$\boldsymbol{\sigma}_{\text{material}} = \mathbf{Q}\boldsymbol{\sigma}_{\text{lab}}\mathbf{Q}^T$$

The **principal stresses** (eigenvalues of $\boldsymbol{\sigma}$) and **invariants** (trace, determinant) are **invariant** under this transformation, which is why yield criteria based on invariants (e.g., von Mises: $f = \sqrt{3J_2} - \sigma_y$) are **isotropic** (independent of orientation).

20.2. A2.2 Detailed Second-Order Tensor Algebra

20.2.1. A2.2.1 Formal Definition

A **second-order tensor A** is a **linear map** from vectors to vectors:

$$\mathbf{A} : V \to V, \quad \mathbf{b} = \mathbf{A} \cdot \mathbf{u}$$

In component form (using any basis):

$$b_i = A_{ij}u^j \quad \text{(in mixed index form)}$$

or

$$b^i = A^i_j u^j, \quad \text{etc.}$$

The key property: under a coordinate transformation (change of basis), the components transform in a specific way that ensures the geometric object $\mathbf{A}$ remains unchanged.

20.2.2. A2.2.2 Dyadic Product as Building Block

The **dyadic (tensor) product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the second-order tensor:

$$\mathbf{a} \otimes \mathbf{b}, \quad (\mathbf{a} \otimes \mathbf{b}) : \mathbf{u} = \mathbf{a}(\mathbf{b} \cdot \mathbf{u})$$

In components (Cartesian basis):

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$$

Every second-order tensor can be written as a sum of dyads:

$$\mathbf{A} = A^{ij} \mathbf{g}_i \otimes \mathbf{g}_j = A_{ij} \mathbf{g}^i \otimes \mathbf{g}^j = A^i_j \mathbf{g}_i \otimes \mathbf{g}^j$$

and so on (four possible mixed/pure forms).

20.2.3. A2.2.3 Tensor Algebra: Operations

Operation	Formula	Result
Addition	$(\mathbf{A} + \mathbf{B})_{ij} = A_{ij} + B_{ij}$	2nd-order tensor
Scalar mult.	$(\alpha \mathbf{A})_{ij} = \alpha A_{ij}$	2nd-order tensor
Composition	$(\mathbf{AB})_{ij} = A_{ik} B_{kj}$	2nd-order tensor (matrix product)
Transpose	$(\mathbf{A}^T)_{ij} = A_{ji}$	2nd-order tensor
Trace	$\mathrm{tr}\, \mathbf{A} = A_{ii}$	Scalar
Determinant	$\det \mathbf{A}$	Scalar (volume ratio)

Operation	Formula	Result
Outer product	$(\mathbf{A} \otimes \mathbf{B})_{ijkl} = A_{ij}B_{kl}$	4th-order tensor
Single contraction	$(\mathbf{A} \cdot \mathbf{B})_{ij} = A_{ik}B_{kj}$	2nd-order tensor
Double contraction	$\mathbf{A} : \mathbf{B} = A_{ij}B_{ij}$	Scalar

20.2.4. A2.2.4 Symmetric and Antisymmetric Decomposition

Any second-order tensor can be uniquely decomposed into **symmetric** and **antisymmetric** parts:

$$\mathbf{A} = \operatorname{sym}\mathbf{A} + \operatorname{skew}\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$$

Physical example: In continuum mechanics, the **velocity gradient** $\mathbf{L} = \nabla\mathbf{v}$ decomposes into: - **Rate of deformation** $\mathbf{D} = \operatorname{sym}\mathbf{L}$ (stretching, volume change) - **Spin tensor** $\mathbf{W} = \operatorname{skew}\mathbf{L}$ (rigid rotation)

A symmetric tensor has **6 independent components** in 3D; an antisymmetric tensor has **3 independent components** (equivalent to a vector via the axial vector).

20.2.5. A2.2.5 Determinant and Volume Ratio

The **determinant** of $\mathbf{A}$ is defined geometrically: it is the signed **volume ratio**:

$$\det\mathbf{A} = \frac{\text{volume of } \{\mathbf{Au}, \mathbf{Av}, \mathbf{Aw}\}}{\text{volume of } \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}}$$

Component form:

$$\det\mathbf{A} = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{lmn}A_{il}A_{jm}A_{kn}$$

or more simply (in index notation):

$$\det \mathbf{A} = e_{ijk} A^i_1 A^j_2 A^k_3 = \det[A^i_j]$$

Properties: - $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$ (multiplicativity) - $\det(\mathbf{A}^T) = \det \mathbf{A}$ - $\det(\alpha \mathbf{A}) = \alpha^3 \det \mathbf{A}$ (in 3D, scales as cube) - $\det \mathbf{A}^{-1} = 1/\det \mathbf{A}$ (if invertible) - For an orthogonal tensor $\mathbf{Q}$, $\det \mathbf{Q} = \pm 1$ (typically $+1$ for proper rotations)

20.2.6. A2.2.6 Eigenvalues, Principal Directions, and Spectral Decomposition

For a **symmetric tensor $\mathbf{A}$**, the eigenvalue problem is:

$$\mathbf{A} \cdot \mathbf{n}^{(\alpha)} = \lambda^{(\alpha)} \mathbf{n}^{(\alpha)}, \quad \alpha = 1, 2, 3$$

Key results (Spectral Theorem for Symmetric Tensors): - All three eigenvalues $\lambda^{(\alpha)}$ are **real**. - The three eigenvectors $\mathbf{n}^{(\alpha)}$ are **orthogonal** (if eigenvalues are distinct). - The tensor is diagonalizable:

$$\mathbf{A} = \sum_{\alpha=1}^{3} \lambda^{(\alpha)} \mathbf{n}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)}$$

This **spectral decomposition** is the basis for tensor functions: any analytic function f applied to $\mathbf{A}$ acts on the eigenvalues:

$$f(\mathbf{A}) = \sum_{\alpha=1}^{3} f(\lambda^{(\alpha)}) \mathbf{n}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)}$$

Examples: $\mathbf{A}^2$, $\sqrt{\mathbf{A}}$, $\ln \mathbf{A}$, $\exp \mathbf{A}$.

20.2.7. A2.2.7 Invariants of a Second-Order Tensor

For a general second-order tensor $\mathbf{A}$, the **principal invariants** are the coefficients of the characteristic polynomial:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0$$

Explicit formulas:

$$I_1 = \operatorname{tr} \mathbf{A} = A_{ii}$$
$$I_2 = \frac{1}{2}[(\operatorname{tr} \mathbf{A})^2 - \operatorname{tr}(\mathbf{A}^2)] = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji})$$
$$I_3 = \det \mathbf{A}$$

Important property: These invariants are **unchanged** under orthogonal coordinate transformations (they depend only on the geometry of $\mathbf{A}$, not the basis).

Deviatoric invariants are also important. Define the **deviatoric part**:

$$\mathbf{A}^{\text{dev}} = \mathbf{A} - \frac{1}{3}(\operatorname{tr} \mathbf{A})\mathbf{I}$$

The second deviatoric invariant is:

$$J_2 = \frac{1}{2}\mathbf{A}^{\text{dev}} : \mathbf{A}^{\text{dev}} = \frac{1}{2}A'_{ij}A'_{ij}$$

This is widely used in plasticity (von Mises criterion: $\sqrt{3J_2} = \sigma_y$).

Worked example: Stress tensor in Cartesian coordinates:

$$\boldsymbol{\sigma} = \begin{pmatrix} 10 & 5 & 0 \\ 5 & 20 & 0 \\ 0 & 0 & 5 \end{pmatrix} \text{ MPa}$$

Invariants:

$$I_1 = 10 + 20 + 5 = 35 \text{ MPa}$$

$$I_2 = \frac{1}{2}(35^2 - (10^2 + 20^2 + 5^2 + 2 \cdot 5^2)) = \frac{1}{2}(1225 - 525) = 350 \text{ MPa}^2$$

$$I_3 = \det \boldsymbol{\sigma} = 10(20 \cdot 5 - 0) - 5(5 \cdot 5 - 0) = 1000 - 125 = 875 \text{ MPa}^3$$

Mean stress: $\sigma_m = I_1/3 \approx 11.67$ MPa.

Deviatoric part:

$$\mathbf{s} = \begin{pmatrix} -1.67 & 5 & 0 \\ 5 & 8.33 & 0 \\ 0 & 0 & -6.67 \end{pmatrix}$$

$$J_2 = \frac{1}{2}(1.67^2{+}8.33^2{+}6.67^2{+}2{\cdot}5^2) = \frac{1}{2}(2.79{+}69.39{+}44.49{+}50) \approx 83.3 \text{ MPa}^2$$

Equivalent von Mises stress: $\sigma_{\text{eq}} = \sqrt{3J_2} \approx 15.8$ MPa.

(If material yields at $\sigma_y = 15.8$ MPa, this stress state is on the yield surface.)

20.3. A2.3 Fourth-Order Tensor Algebra

20.3.1. A2.3.1 Definition and General Form

A **fourth-order tensor** $\mathbb{A}$ (denoted in blackboard bold, see A01) is a linear map from second-order tensors to second-order tensors:

$$\mathbf{B} = \mathbb{A} : \mathbf{A} \quad \text{(double contraction)}$$

In components:

$$B_{ij} = A_{ijkl}C_{kl}$$

The general form in basis tensors is:

$$\mathbb{A} = A^{ijkl}\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k \otimes \mathbf{g}_l$$

with 81 components in 3D. The double contraction is:

$$\mathbb{A} : \mathbf{C} = A^{ijkl}C_{kl}\mathbf{g}_i \otimes \mathbf{g}_j$$

20.3.2. A2.3.2 Fourth-Order Identity Tensor

The **fourth-order identity tensor** $\mathbb{I}$ satisfies:

$$\mathbb{I} : \mathbf{A} = \mathbf{A} \quad \text{for all 2nd-order tensors } \mathbf{A}$$

Component form (in Cartesian basis):

$$I_{ijkl} = \delta_{ik}\delta_{jl}$$

Verify: $(\mathbb{I} : \mathbf{A})_{ij} = I_{ijkl}A_{kl} = \delta_{ik}\delta_{jl}A_{kl} = A_{ij}$ ✓

Major transpose: Swapping the first pair and second pair of indices gives:

$$\mathbb{I}^T = J \quad \text{with} \quad J_{ijkl} = \delta_{il}\delta_{jk}$$

The **symmetric identity** (used in isotropic elasticity) is:

$$\mathbb{I}^s = \frac{1}{2}(\mathbb{I} + \mathbb{I}^T) \quad \text{with} \quad I^s_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

20.3.3. A2.3.3 Symmetries and Constraints

A fourth-order tensor may have **symmetries** that reduce its number of independent components.

Major symmetry: $\mathbb{A}_{ijkl} = \mathbb{A}_{klij}$ (invariant under swapping pairs), reduces to 36 independent components.

Minor symmetries: - $\mathbb{A}_{ijkl} = \mathbb{A}_{jikl}$ (symmetric in first pair) - $\mathbb{A}_{ijkl} = \mathbb{A}_{ijlk}$ (symmetric in second pair)

If all three symmetries hold (major + both minors), there are **21 independent components**. This is the case for the **elastic stiffness tensor** $\mathbb{C}$ in linear elasticity, because stress and strain are symmetric.

Isotropic elasticity: Further reduced to **2 independent parameters** (λ and μ, the Lamé constants).

20.3.4. A2.3.4 Deviatoric Projector

The **deviatoric projector** extracts the trace-free part of a tensor:

$$\mathbb{P}_{\text{dev}} = \mathbb{I}^s - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}$$

In component form:

$$(P_{\text{dev}})_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}$$

Applied to a tensor: $\mathbf{A}^{\text{dev}} = \mathbb{P}_{\text{dev}} : \mathbf{A}$ removes the trace:

$$\mathbf{A}^{\text{dev}} = \mathbf{A} - \frac{1}{3}(\operatorname{tr}\mathbf{A})\mathbf{I}$$

Use: Decomposing stress into hydrostatic and deviatoric parts for isotropic yield criteria.

20.3.5. A2.3.5 Voigt Notation for Matrices

In numerical implementations, fourth-order tensors with symmetries are stored as **6×6 matrices** using **Voigt notation**, which maps the 9 components of a symmetric 2nd-order tensor to a 6-component vector:

$$\sigma_{11} \to \sigma_1, \quad \sigma_{22} \to \sigma_2, \quad \sigma_{33} \to \sigma_3, \sigma_{23} = \sigma_{32} \to \sigma_4, \quad \sigma_{13} = \sigma_{31} \to \sigma_5, \quad \sigma_{12} = \sigma_{21} \to \sigma_6$$

Similarly for strain $\boldsymbol{\varepsilon}$ (but note: engineering shear strains $\gamma_{ij} = 2\varepsilon_{ij}$ are used, so Voigt is **not** an isometry).

The elasticity tensor $\mathbb{C}^e$ (fourth-order, with major and minor symmetries) becomes a **6×6 matrix $\mathbf{C}^{\text{Voigt}}$**:

$$\sigma_i = C_{ij}^{\text{Voigt}} \varepsilon_j$$

For isotropic elasticity:

$$\mathbf{C}^{\text{Voigt}} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

where μ (shear modulus) and κ (bulk modulus) are related by $\lambda = 3\kappa - 2\mu$.

20.3.6. A2.3.6 Component Transformation

Under an orthogonal coordinate transformation with rotation matrix $\mathbf{Q}$, fourth-order tensor components transform as:

$$\bar{A}_{ijkl} = Q_{im} Q_{jn} Q_{kp} Q_{lq} A_{mnpq}$$

This ensures that the geometric object remains invariant.

20.3.7. A2.3.7 Connection to Material Stiffness

In linear elasticity, the **stress-strain relationship** is:

$$\boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon}$$

where $\mathbb{C}^e$ is the **fourth-order elasticity tensor** with components:

$$\sigma_{ij} = C^e_{ijkl}\varepsilon_{kl}$$

For **isotropic materials**:

$$C^e_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where λ is Lamé's first parameter and μ is the shear modulus.

The inverse relationship (compliance) is:

$$\boldsymbol{\varepsilon} = \mathbb{S} : \boldsymbol{\sigma}, \quad S_{ijkl} = \frac{1}{2\mu}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{\lambda}{6\mu(\lambda+\mu)}\delta_{ij}\delta_{kl}$$

In finite-strain hyperelasticity (L05), the material tangent stiffness is:

$$\mathbb{D}_{ijkl} = \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}}$$

where $W(\mathbf{E})$ is the strain energy density.

20.3.8. A2.3.8 Inverse of a Fourth-Order Tensor

If a fourth-order tensor $\mathbb{A}$ is invertible, its inverse $\mathbb{A}^{-1}$ satisfies:

$$\mathbb{A}^{-1} : \mathbb{A} = \mathbb{I}, \quad \mathbb{A} : \mathbb{A}^{-1} = \mathbb{I}$$

For the elasticity tensor, the inverse is the compliance tensor:

$$\mathbb{C}^e : \mathbb{S} = \mathbb{I}$$

Computing the inverse in full tensor form is tedious; using Voigt notation (6×6 matrix inversion) is standard in FEM codes.

20.4. Summary and References to Other Appendices

This appendix complements: - **A01 (Notation):** All symbols and typography rules - **L01 (Math Foundations):** Definitions and basic properties - **L03 (Kinematics):** Applications of tensors to deformation analysis - **L07 (Plasticity):** Fourth-order tensors in constitutive modeling

For further reading on tensor calculus and differential geometry, consult the course bibliography or texts like *Bonet and Wood* (Nonlinear Continuum Mechanics for Finite Element Analysis) or *Marsden & Hughes* (Mathematical Foundations of Elasticity).

21. Appendix A — Skew Tensors, Permutation, and Tensor Algebra

Axial vectors, the Levi-Civita tensor, and applications

This appendix gathers the algebra and advanced properties of skew-symmetric 2nd-order tensors and the permutation tensor, which underpin many formulas in continuum mechanics and elasticity. References to L02 tensor calculus appear throughout.

21.1. A3.1 Skew-Symmetric Tensors and Axial Vectors

21.1.1. Definition and Structure

A second-order tensor $\mathbf{W}$ is **skew-symmetric** (antisymmetric) if:

$$\mathbf{W}^T = -\mathbf{W}, \quad \text{or equivalently} \quad W_{ij} = -W_{ji}$$

In 3D, a skew-symmetric tensor has exactly **three independent components**. By contrast, a general 2nd-order tensor has 9 components, and a symmetric tensor has 6 components.

Example: The spin tensor (skew part of the velocity gradient):

$$\mathbf{W} = \text{skew}(\mathbf{L}) = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)$$

21.1.2. The Axial Vector

Every skew-symmetric tensor $\mathbf{W}$ in 3D corresponds to a unique **axial vector $\mathbf{w}$** (also called the **dual vector** or **pseudovector**) such that:

$$\mathbf{W}\mathbf{v} = \mathbf{w} \times \mathbf{v} \quad \forall\, \mathbf{v}$$

Extraction formula: Given $\mathbf{W}$, extract $\mathbf{w}$ by:

$$w_i = -\frac{1}{2}\varepsilon_{ijk}W_{jk}$$

where ε_{ijk} is the **Levi-Civita symbol** (defined in Section A3.2 below).

Inverse (construction formula): Given $\mathbf{w}$, construct the skew tensor:

$$W_{ij} = -\varepsilon_{ijk}w_k = \varepsilon_{ikj}w_k$$

21.1.3. Component Relation

In component form, with $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$:

$$\mathbf{W} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

Check: $\mathbf{W}\mathbf{v} = \begin{pmatrix} -w_3v_2 + w_2v_3 \\ w_3v_1 - w_1v_3 \\ -w_2v_1 + w_1v_2 \end{pmatrix} = \mathbf{w} \times \mathbf{v}$ ✓

21.1.4. Physical Interpretation: Angular Velocity

In kinematics, the spin tensor $\mathbf{W} = \text{skew}(\dot{\mathbf{F}}\mathbf{F}^{-1})$ has axial vector $\mathbf{w} = \boldsymbol{\omega}$, the **angular velocity** of the material.

A material point undergoing rigid rotation with angular velocity $\boldsymbol{\omega}$ has velocity:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$$

Equivalently (in matrix form):

$$\mathbf{v} = \mathbf{W}\mathbf{x}, \quad \mathbf{W} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

21.1.5. Worked Example: Extracting Angular Velocity

Suppose a rigid body has spin tensor:

$$\mathbf{W} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Extract the axial vector using $w_i = -\frac{1}{2}\varepsilon_{ijk}W_{jk}$:

- $w_1 = -\frac{1}{2}(\varepsilon_{123}W_{23} + \varepsilon_{132}W_{32}) = -\frac{1}{2}(1 \cdot 0 + (-1) \cdot 0) = 0$
- $w_2 = -\frac{1}{2}(\varepsilon_{231}W_{31} + \varepsilon_{213}W_{13}) = -\frac{1}{2}(1 \cdot 2 + (-1) \cdot (-2)) = -\frac{1}{2}(2+2) = -2$
- $w_3 = -\frac{1}{2}(\varepsilon_{312}W_{12} + \varepsilon_{321}W_{21}) = -\frac{1}{2}(1 \cdot 0 + (-1) \cdot 0) = 0$

So $\mathbf{w} = (0, -2, 0)^T$, meaning the body rotates about the $\mathbf{e}_2$-axis with angular velocity magnitude 2 rad/s.

21.2. A3.2 The Permutation Tensor and Levi-Civita Symbol

21.2.1. Levi-Civita Symbol (Notation)

The **Levi-Civita symbol** ε_{ijk} is a **coordinate-dependent symbol** with values:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if any two indices are equal} \end{cases}$$

Explicit values (in 3D):

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \quad \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1, \quad \text{all others } = 0$$

21.2.2. Levi-Civita Tensor (True Tensor)

The **Levi-Civita tensor $\boldsymbol{\varepsilon}$** is a **genuine third-order tensor** that transforms properly under coordinate changes. In Cartesian coordinates, its components coincide with the symbol:

$$\varepsilon_{ijk}^{\text{Cartesian}} = \varepsilon_{ijk}^{\text{symbol}}$$

However, in curvilinear coordinates with metric tensor g_{ij}, the tensor components are:

$$\varepsilon_{ijk}^{\text{tensor}} = \sqrt{g}\,\varepsilon_{ijk}^{\text{symbol}}, \quad g = \det(g_{ij})$$

This course uses **Cartesian coordinates exclusively**, so we work with the symbol and tensor interchangeably.

21.2.3. Epsilon-Delta Identity

The most useful formula relates the permutation symbol to the Kronecker delta:

$$\boxed{\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}}$$

Proof sketch: Expand both sides for all values of i, j, l, m and verify that both equal 1 when $\{i, j\} = \{l, m\}$ (in the same or opposite order) and 0 otherwise. ✓

Special cases: - $\varepsilon_{ijk}\varepsilon_{ijk} = 6$ (contracting all three pairs) - $\varepsilon_{ijk}\varepsilon_{ljk} = 2\delta_{il}$ (contracting the last two pairs)

21.2.4. Determinant Formula

The determinant of a 3×3 matrix $\mathbf{A}$ can be written as:

$$\det \mathbf{A} = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3} = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k}$$

More generally, any column can be used:

$$\det \mathbf{A} = \varepsilon_{ijk}A_{im}A_{jn}A_{kp}\varepsilon_{mnp}$$

Application: In continuum mechanics, the volume change is $J = \det \mathbf{F} = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{lmn}F_{il}F_{jm}F_{kn}$.

21.2.5. Vector Triple Product

The **vector triple product** $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ can be expanded using permutation notation:

$$[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]_i = \varepsilon_{ijk}a_j(\mathbf{b}\times\mathbf{c})_k = \varepsilon_{ijk}a_j\varepsilon_{klm}b_lc_m$$

Using the epsilon-delta identity with $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$:

$$[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]_i = a_jb_ic_j - a_jc_ib_j = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$\boxed{\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}}$$

This formula is invaluable in deriving vector identities and checking consistency in mechanics equations.

21.2.6. Curl (Rotation) Operator

In index notation, the curl of a vector field $\mathbf{u}(\mathbf{x})$ is:

$$(\nabla\times\mathbf{u})_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

For example, the vorticity in fluid mechanics is:

$$\boldsymbol{\omega} = \nabla\times\mathbf{v} = \text{axial vector of skew}(\nabla\mathbf{v})$$

The curl is **automatically skew** because ε_{ijk} is skew in its last two indices.

21.3. A3.3 Lie Algebra of Skew-Symmetric Tensors: $\mathfrak{so}(3)$

21.3.1. The Lie Algebra $\mathfrak{so}(3)$

The set of all 3×3 skew-symmetric tensors forms a **Lie algebra** denoted $\mathfrak{so}(3)$, with the commutator as the Lie bracket:

$$[\mathbf{W}_1, \mathbf{W}_2] = \mathbf{W}_1\mathbf{W}_2 - \mathbf{W}_2\mathbf{W}_1$$

Properties: - **Closure:** The commutator of two skew tensors is skew. - **Dimension:** 3 (one basis element per axial-vector component). - **Basis:** Three generators corresponding to rotations about each axis:

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{E}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A general skew tensor $\mathbf{W}$ with axial vector $\mathbf{w} = (w_1, w_2, w_3)$ is:

$$\mathbf{W} = w_1\mathbf{E}_1 + w_2\mathbf{E}_2 + w_3\mathbf{E}_3$$

21.3.2. The Exponential Map: From Skew to Rotation

The **matrix exponential** of a skew tensor $\mathbf{W}$ gives a rotation:

$$\mathbf{R}(t) = \exp(t\mathbf{W}) = \mathbf{I} + t\mathbf{W} + \frac{t^2}{2!}\mathbf{W}^2 + \frac{t^3}{3!}\mathbf{W}^3 + \cdots$$

is an orthogonal matrix with $\det \mathbf{R} = 1$ (a **rotation**).

21.3.3. Rodrigues' Formula

For a unit axial vector $\hat{\mathbf{w}}$ (with magnitude 1) and a scalar angle θ, let:

$$\mathbf{W}_\theta = \theta \text{skew}(\hat{\mathbf{w}})$$

Then the exponential closes in closed form (Rodrigues' rotation formula):

$$\boxed{\mathbf{R} = \exp(\mathbf{W}_\theta) = \mathbf{I} + \sin\theta\, \mathbf{W}_\theta + (1 - \cos\theta)\mathbf{W}_\theta^2}$$

This describes a **rotation by angle θ about the axis $\hat{\mathbf{w}}$**.

Key property: For any skew tensor $\mathbf{W}$, the exponential $\mathbf{R} = \exp(\mathbf{W})$ is a rotation tensor. Conversely, any rotation can be written as the exponential of a (unique) skew tensor.

21.3.4. Connection to Kinematics

When the spin tensor $\mathbf{W} = \text{skew}(\mathbf{L})$ is given, the rotation part of the deformation gradient evolves as:

$$\frac{d\mathbf{R}}{dt} = \mathbf{W}\mathbf{R}$$

Integrating formally (if $\mathbf{W}$ is constant over a time interval $[0, t]$):

$$\mathbf{R}(t) = \exp(\mathbf{W}t)\mathbf{R}(0)$$

In a finite-element or finite-difference code, this is often integrated using Rodrigues' formula or similar exponential-map algorithms to maintain orthogonality of $\mathbf{R}$ exactly.

21.4. A3.4 Summary of Key Formulas

Reference: Permutation and Skew-Tensor Identities

Axial Vector:

$$w_i = -\frac{1}{2}\varepsilon_{ijk}W_{jk}, \quad W_{ij} = -\varepsilon_{ijk}w_k$$

Epsilon-Delta:

$$\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Cross Product (Component Form):

$$(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk}a_j b_k$$

Vector Triple Product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Determinant (Column 1):

$$\det \mathbf{A} = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$$

Rodrigues' Formula:

$$\mathbf{R} = \mathbf{I} + \sin\theta\,\mathbf{W} + (1 - \cos\theta)\mathbf{W}^2, \quad \mathbf{W} = \mathrm{skew}(\hat{\mathbf{w}})$$

All formulas assume **Cartesian coordinates** and 3D Euclidean space.

22. L08 Appendix — Worked Examples

Combined Hardening: RRM with Linear Kinematic + Power-Law Isotropic

22.1. Purpose

L08 presents the radial return method (RRM) for J2 plasticity with **linear isotropic** hardening, where the plastic multiplier update has a closed form. This appendix carries out the full derivation for a richer combined-hardening model — **linear kinematic (Prager)** plus **power-law isotropic** — where the update becomes a non-linear scalar equation and a local Newton solve is required. Every step is written out so the derivation can be followed without prior knowledge beyond L08. Brief recap of standard material is included where it aids the flow; see L08 for the underlying theory.

The running conventions are the same as L08: $\boldsymbol{\varepsilon}$ for total strain, $\boldsymbol{\varepsilon}^p$ for plastic strain, $\bar{\varepsilon}^p$ for equivalent plastic strain, $\boldsymbol{\sigma}$ for Cauchy stress, $\mathbf{s} = \mathrm{dev}(\boldsymbol{\sigma})$ for the deviatoric stress, $\boldsymbol{\alpha}$ for the backstress, $\Delta\gamma$ for the discrete plastic multiplier, $\mathbb{C}^e$ for the fourth-order elasticity tensor, and μ, κ for the shear and bulk moduli.

22.2. Model Definition

Yield function (combined hardening, J2):

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \bar{\varepsilon}^p) = \|\mathbf{s} - \boldsymbol{\alpha}\| - \sqrt{\tfrac{2}{3}}\,\sigma_y(\bar{\varepsilon}^p) \;\le\; 0,$$

where

- $\mathbf{s} = \mathrm{dev}(\boldsymbol{\sigma})$ is the deviatoric Cauchy stress;
- $\boldsymbol{\alpha}$ is the backstress tensor, deviatoric in J2 plasticity, marking the centre of the yield surface in deviatoric stress space;
- $\bar{\varepsilon}^p$ is the scalar equivalent plastic strain (isotropic hardening variable);
- $\|\cdot\|$ denotes the Frobenius norm, $\|\mathbf{X}\| = \sqrt{\mathbf{X} : \mathbf{X}}$;
- $\sigma_y(\bar{\varepsilon}^p)$ is the current yield stress.

Isotropic hardening — power law:

$$\sigma_y(\bar{\varepsilon}^p) \;=\; \sigma_{y0} \;+\; H_{\mathrm{iso}}\,(\bar{\varepsilon}^p)^m,$$

with initial yield σ_{y0}, isotropic hardening coefficient H_{iso}, and exponent $m \in [0, 1]$.

Kinematic hardening — linear Prager rule:

$$\dot{\boldsymbol{\alpha}} \;=\; H_{\mathrm{kin}}\,\dot{\boldsymbol{\varepsilon}}^p \quad\Longrightarrow\quad \Delta\boldsymbol{\alpha} \;=\; H_{\mathrm{kin}}\,\Delta\boldsymbol{\varepsilon}^p,$$

with constant scalar kinematic hardening modulus H_{kin}. In J2 plasticity $\dot{\boldsymbol{\varepsilon}}^p$ is deviatoric, so $\dot{\boldsymbol{\alpha}}$ is deviatoric; if $\boldsymbol{\alpha}_0$ is deviatoric, $\boldsymbol{\alpha}$ remains deviatoric for all time.

Effective (shifted) stress. Introduce

$$\boldsymbol{\xi} \;=\; \boldsymbol{\sigma} - \boldsymbol{\alpha}, \qquad \mathbf{s}_\xi \;=\; \mathbf{s} - \boldsymbol{\alpha}$$

(the second identity uses that $\boldsymbol{\alpha}$ is deviatoric). The yield function rewrites as

$$f(\mathbf{s}_\xi, \bar{\varepsilon}^p) \;=\; \|\mathbf{s}_\xi\| - \sqrt{\tfrac{2}{3}}\,\sigma_y(\bar{\varepsilon}^p) \;\le\; 0,$$

which makes the algorithm essentially a radial return in $\mathbf{s}_\xi$-space.

22.3. Step 1 — Elastic Predictor

Given the state at t_n, namely $\boldsymbol{\sigma}_n,\ \boldsymbol{\varepsilon}^p_n,\ \boldsymbol{\alpha}_n,\ \bar{\varepsilon}^p_n$, and the prescribed total strain increment $\Delta\boldsymbol{\varepsilon}$, freeze plastic flow and form the trial state (denoted by a "tr" superscript):

1. **Trial stress:**
$$\boldsymbol{\sigma}^{\mathrm{tr}}_{n+1} \;=\; \boldsymbol{\sigma}_n + \mathbb{C}^e : \Delta\boldsymbol{\varepsilon}.$$
2. **Trial backstress** (unchanged over an elastic step):
$$\boldsymbol{\alpha}^{\mathrm{tr}}_{n+1} \;=\; \boldsymbol{\alpha}_n.$$
3. **Trial equivalent plastic strain** (unchanged):
$$\bar{\varepsilon}^{p,\mathrm{tr}}_{n+1} \;=\; \bar{\varepsilon}^p_n.$$
4. **Trial deviatoric effective stress:**
$$\mathbf{s}^{\mathrm{tr}}_\xi \;=\; \mathrm{dev}(\boldsymbol{\sigma}^{\mathrm{tr}}_{n+1}) - \boldsymbol{\alpha}^{\mathrm{tr}}_{n+1}.$$

22.4. Step 2 — Yield Check

Evaluate the yield function at the trial state:

$$f^{\mathrm{tr}} \;=\; \|\mathbf{s}^{\mathrm{tr}}_\xi\| - \sqrt{\tfrac{2}{3}}\,\sigma_y(\bar{\varepsilon}^{p,\mathrm{tr}}_{n+1}) \;=\; \|\mathbf{s}^{\mathrm{tr}}_\xi\| - \sqrt{\tfrac{2}{3}}\,\big(\sigma_{y0} + H_{\mathrm{iso}}\,(\bar{\varepsilon}^p_n)^m\big).$$

- **If $f^{\mathbf{tr}} \le$ TOL:** the step is elastic. Accept the trial state,

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}^{\mathrm{tr}}_{n+1}, \quad \boldsymbol{\varepsilon}^p_{n+1} = \boldsymbol{\varepsilon}^p_n, \quad \boldsymbol{\alpha}_{n+1} = \boldsymbol{\alpha}_n, \quad \bar{\varepsilon}^p_{n+1} = \bar{\varepsilon}^p_n,$$

  and exit.
- **If $f^{\mathbf{tr}} >$ TOL:** plastic yielding occurs — proceed to Step 3.

22.5. Step 3 — Plastic Corrector: Core Equations

Flow rule (associative, J2):

$$\Delta\boldsymbol{\varepsilon}^p \;=\; \Delta\gamma\,\frac{\partial f}{\partial \boldsymbol{\sigma}} \;=\; \Delta\gamma\,\frac{\mathbf{s}_{\xi,n+1}}{\|\mathbf{s}_{\xi,n+1}\|}, \qquad \Delta\gamma \ge 0.$$

Isotropic hardening update:

$$\bar{\varepsilon}^p_{n+1} \;=\; \bar{\varepsilon}^p_n + \sqrt{\tfrac{2}{3}}\,\Delta\gamma.$$

Kinematic hardening update:

$$\boldsymbol{\alpha}_{n+1} \;=\; \boldsymbol{\alpha}_n + H_{\mathrm{kin}}\,\Delta\boldsymbol{\varepsilon}^p.$$

Stress update:

$$\boldsymbol{\sigma}_{n+1} \;=\; \boldsymbol{\sigma}^{\mathrm{tr}}_{n+1} - \mathbb{C}^e : \Delta\boldsymbol{\varepsilon}^p.$$

Deviatoric effective stress update. Expand $\mathbf{s}_{\xi,n+1} = \mathrm{dev}(\boldsymbol{\sigma}_{n+1}) - \boldsymbol{\alpha}_{n+1}$ and substitute the two update equations above:

$$\mathbf{s}_{\xi,n+1} \;=\; \big(\mathrm{dev}(\boldsymbol{\sigma}^{\mathrm{tr}}_{n+1}) - 2\mu\,\Delta\boldsymbol{\varepsilon}^p\big) - \big(\boldsymbol{\alpha}_n + H_{\mathrm{kin}}\,\Delta\boldsymbol{\varepsilon}^p\big).$$

Here $\mathbb{C}^e : \Delta\boldsymbol{\varepsilon}^p = 2\mu\,\Delta\boldsymbol{\varepsilon}^p$ because $\Delta\boldsymbol{\varepsilon}^p$ is deviatoric (so the volumetric $\kappa\,\mathrm{tr}(\Delta\boldsymbol{\varepsilon}^p)\,\mathbf{I}$ term vanishes). Rearranging gives

$$\mathbf{s}_{\xi,n+1} \;=\; \mathbf{s}^{\mathrm{tr}}_{\xi} - (2\mu + H_{\mathrm{kin}})\,\Delta\boldsymbol{\varepsilon}^p,$$

and substituting the flow rule for $\Delta\boldsymbol{\varepsilon}^p$:

$$\mathbf{s}_{\xi,n+1} \;=\; \mathbf{s}_\xi^{\mathrm{tr}} - (2\mu + H_{\mathrm{kin}})\,\Delta\gamma\,\frac{\mathbf{s}_{\xi,n+1}}{\|\mathbf{s}_{\xi,n+1}\|}.$$

Radial return in effective-stress space. The previous equation shows that $\mathbf{s}_{\xi,n+1}$ is a positive scalar multiple of $\mathbf{s}_\xi^{\mathrm{tr}}$, so they share a common direction:

$$\frac{\mathbf{s}_{\xi,n+1}}{\|\mathbf{s}_{\xi,n+1}\|} \;=\; \frac{\mathbf{s}_\xi^{\mathrm{tr}}}{\|\mathbf{s}_\xi^{\mathrm{tr}}\|} \;\equiv\; \mathbf{n}_\xi^{\mathrm{tr}}.$$

Taking norms,

$$\|\mathbf{s}_{\xi,n+1}\| \;=\; \|\mathbf{s}_\xi^{\mathrm{tr}}\| - (2\mu + H_{\mathrm{kin}})\,\Delta\gamma. \tag{$\star$}$$

This is the "radial return" relation for the effective deviatoric stress.

22.6. Step 4 — Solving for the Plastic Multiplier $\Delta\gamma$

Consistency at t_{n+1} requires $f(\mathbf{s}_{\xi,n+1}, \bar{\varepsilon}^p_{n+1}) = 0$:

$$\|\mathbf{s}_{\xi,n+1}\| \;=\; \sqrt{\tfrac{2}{3}}\,\sigma_y(\bar{\varepsilon}^p_{n+1}) \;=\; \sqrt{\tfrac{2}{3}}\left(\sigma_{y0} + H_{\mathrm{iso}}\left(\bar{\varepsilon}^p_n + \sqrt{\tfrac{2}{3}}\,\Delta\gamma\right)^m\right).$$

Equating this with ($\star$) and bringing everything onto one side gives a **non-linear scalar equation in $\Delta\gamma$** (non-linear because of the exponent m):

$$R(\Delta\gamma) \;=\; (2\mu{+}H_{\mathrm{kin}})\,\Delta\gamma + \sqrt{\tfrac{2}{3}}\left(\sigma_{y0}{+}H_{\mathrm{iso}}\left(\bar{\varepsilon}^p_n{+}\sqrt{\tfrac{2}{3}}\,\Delta\gamma\right)^m\right) - \|\mathbf{s}_\xi^{\mathrm{tr}}\| \;=\; 0.$$

The trial overshoot $f^{\mathrm{tr}} = \|\mathbf{s}_\xi^{\mathrm{tr}}\| - \sqrt{2/3}\,\sigma_y(\bar{\varepsilon}^p_n)$ is the "gap" the corrector must close; in particular $R(0) = -f^{\mathrm{tr}} < 0$ and R is monotonically increasing in $\Delta\gamma$ for $m \in [0,1]$, so there is a unique root $\Delta\gamma > 0$.

Local Newton–Raphson. At iterate j,

$$\Delta\gamma^{(j+1)} \;=\; \Delta\gamma^{(j)} - \frac{R(\Delta\gamma^{(j)})}{R'(\Delta\gamma^{(j)})},$$

with derivative

$$\begin{aligned} R'(\Delta\gamma) \;&=\; (2\mu + H_{\mathrm{kin}}) \;+\; \sqrt{\tfrac{2}{3}}\, H_{\mathrm{iso}}\, m \,\big(\bar{\varepsilon}^p_n + \sqrt{\tfrac{2}{3}}\,\Delta\gamma\big)^{m-1} \cdot \sqrt{\tfrac{2}{3}} \\ &=\; (2\mu + H_{\mathrm{kin}}) \;+\; \tfrac{2}{3}\, H_{\mathrm{iso}}\, m \,\big(\bar{\varepsilon}^p_n + \sqrt{\tfrac{2}{3}}\,\Delta\gamma\big)^{m-1}. \end{aligned}$$

Practical notes:

- A good **initial guess** is $\Delta\gamma^{(0)} = f^{\mathrm{tr}}/(2\mu + H_{\mathrm{kin}} + \tfrac{2}{3} H_{\mathrm{iso}}\, m\,(\bar{\varepsilon}^p_n)^{m-1})$ — this is the first Newton step from zero with the argument frozen. A simpler fallback is $\Delta\gamma^{(0)} = 0$.
- **Stopping criterion:** iterate until $|R(\Delta\gamma^{(j+1)})| < \mathrm{TOL}_R$ or $|\Delta\gamma^{(j+1)} - \Delta\gamma^{(j)}| < \mathrm{TOL}_\gamma$.
- **Edge cases.** If $m = 0$ the isotropic term is constant (perfectly-plastic-isotropic with kinematic hardening still active), R is linear in $\Delta\gamma$, and the closed-form solution is $\Delta\gamma = f^{\mathrm{tr}}/(2\mu + H_{\mathrm{kin}})$. If $m < 1$ and $\bar{\varepsilon}^p_n = 0$, the derivative R' diverges at $\Delta\gamma \to 0$ — regularise the first iterate (e.g. replace $\bar{\varepsilon}^p_n$ by $\max(\bar{\varepsilon}^p_n, \varepsilon_{\mathrm{reg}})$ with $\varepsilon_{\mathrm{reg}} \sim 10^{-12}$) or switch to a small-$\bar{\varepsilon}^p$ closed-form starter.
- Because R is monotone and convex-enough on $[0, \Delta\gamma^\star]$ for $m \in [0, 1]$, Newton converges quadratically from $\Delta\gamma^{(0)} = 0$ in practice.

22.7. Step 5 — Final Updates

Once $\Delta\gamma$ has converged to within tolerance:

1. **Equivalent plastic strain:**

$$\bar{\varepsilon}^p_{n+1} \;=\; \bar{\varepsilon}^p_n + \sqrt{\tfrac{2}{3}}\,\Delta\gamma.$$

2. **Flow direction and plastic-strain increment:**

$$\mathbf{n}_\xi^{\mathrm{tr}} \;=\; \frac{\mathbf{s}_\xi^{\mathrm{tr}}}{\|\mathbf{s}_\xi^{\mathrm{tr}}\|}, \qquad \Delta\boldsymbol{\varepsilon}^p \;=\; \Delta\gamma\,\mathbf{n}_\xi^{\mathrm{tr}}.$$

   (Well-defined because $f^{\mathrm{tr}} > 0$ implies $\|\mathbf{s}_\xi^{\mathrm{tr}}\| \neq 0$.)
3. **Plastic strain tensor:**

$$\boldsymbol{\varepsilon}_{n+1}^p \;=\; \boldsymbol{\varepsilon}_n^p + \Delta\boldsymbol{\varepsilon}^p.$$

4. **Backstress:**

$$\boldsymbol{\alpha}_{n+1} \;=\; \boldsymbol{\alpha}_n + H_{\mathrm{kin}}\,\Delta\boldsymbol{\varepsilon}^p.$$

5. **Cauchy stress:**

$$\boldsymbol{\sigma}_{n+1} \;=\; \boldsymbol{\sigma}_{n+1}^{\mathrm{tr}} - \mathbb{C}^e : \Delta\boldsymbol{\varepsilon}^p \;=\; \boldsymbol{\sigma}_{n+1}^{\mathrm{tr}} - 2\mu\,\Delta\boldsymbol{\varepsilon}^p,$$

   where the second form again uses that $\Delta\boldsymbol{\varepsilon}^p$ is deviatoric, so $\mathrm{tr}(\boldsymbol{\sigma}_{n+1}) = \mathrm{tr}(\boldsymbol{\sigma}_{n+1}^{\mathrm{tr}})$ (the hydrostatic part is preserved by the corrector).

22.8. Consistent (Algorithmic) Tangent — Outline

The consistent tangent $\mathbb{C}_{n+1}^{\mathrm{alg}} = d\boldsymbol{\sigma}_{n+1}/d\boldsymbol{\varepsilon}_{n+1}$ for this combined-hardening model is notably more involved than the linear-hardening J2 case handled in L08. Two new complications appear:

- The power-law isotropic term $H_{\mathrm{iso}}\,(\bar{\varepsilon}^p)^m$ makes $\Delta\gamma$ the solution of a non-linear equation, so $d(\Delta\gamma)/d(\Delta\boldsymbol{\varepsilon})$ must be obtained by **implicit differentiation** of $R(\Delta\gamma) = 0$.
- The kinematic term adds a contribution $H_{\mathrm{kin}}\,\Delta\boldsymbol{\varepsilon}^p$ to the effective-stress update, which itself depends on $\Delta\gamma$.

Option A — analytical tangent via implicit differentiation. Differentiate every equation in Steps 3–5 with respect to $\Delta\boldsymbol{\varepsilon}$ and eliminate intermediate differentials using $R'(\Delta\gamma)$ derived above. The resulting $\mathbb{C}_{n+1}^{\mathrm{alg}}$

retains the same ($\kappa\,\mathbf{I}\otimes\mathbf{I},\ \mathbb{P}_{\rm dev},\ \mathbf{n}\otimes\mathbf{n}$) structure as the J2-linear case in L08, but the scalar coefficients θ_1,θ_2 are modified by $H_{\rm kin}$ and by the effective hardening slope $\frac{2}{3}\,H_{\rm iso}\,m\,(\bar\varepsilon^p_{n+1})^{m-1}$. The normal $\mathbf{n}$ is evaluated at $\mathbf{s}^{\rm tr}_\xi$ (the return is radial in effective-stress space).

Option B — numerical tangent. Perturb each independent component of $\boldsymbol{\varepsilon}_{n+1}$ in turn, rerun the stress update, and form $\mathbb{C}^{\rm alg}_{n+1}$ by finite differences. More expensive (six stress updates per Gauss point in 3D) but robust; useful during model development and for cross-checking analytical derivations.

Either way, the computed $\mathbb{C}^{\rm alg}_{n+1}$ must be used (not the continuum $\mathbb{C}^{ep}$) for the global Newton–Raphson assembly in FEM — otherwise quadratic convergence is lost. See the "Consistent (Algorithmic) Tangent" section of L08 for the underlying argument.

22.9. Suggested Self-Study Exercises

1. **Recover linear isotropic hardening.** Set $H_{\rm kin}=0$ and $m=1$ and show that $R(\Delta\gamma)=0$ reduces to the closed form $\Delta\gamma = f^{\rm tr}/(2\mu+\frac{2}{3}H_{\rm iso})$ used in L08.
2. **Recover pure kinematic hardening.** Set $H_{\rm iso}=0$ and show that R becomes linear in $\Delta\gamma$; derive the closed form.
3. **Implement the local Newton solve for $m=1/2$.** Pick representative values ($\mu=80$ GPa, $\sigma_{y0}=250$ MPa, $H_{\rm iso}=500$ MPa, $H_{\rm kin}=1$ GPa, $\bar\varepsilon^p_n=0.01$) and a trial state with $f^{\rm tr}/\sigma_{y0}\in\{0.01,\ 0.1,\ 1\}$. Plot $R(\Delta\gamma)$ and report the number of Newton iterations to reach $|R|<10^{-10}$.
4. **Derive the analytical consistent tangent.** Follow Option A above to obtain $\mathbb{C}^{\rm alg}_{n+1}$ in closed form. Compare against the numerical tangent of Option B (e.g., central differences with perturbation 10^{-6}) and quantify the component-wise error.

5. **Extend to Armstrong–Frederick kinematic hardening.** Replace the Prager rule with

$$\dot{\boldsymbol{\alpha}} \;=\; \tfrac{2}{3}\, C\, \dot{\boldsymbol{\varepsilon}}^p - \gamma_{\mathrm{AF}}\, \boldsymbol{\alpha}\, \dot{\bar{\varepsilon}}^p.$$

What changes in Steps 3 and 4? Does the return remain radial in effective-stress space? What form does $R(\Delta\gamma)$ take now?